B. Tech. CSE I SEMESTER

PROBABILITY THEORY AND NUMERICAL METHODS

Course Code: A3004

SYLLABUS

UNIT-I (8 Lectures)
PROBABILITY: Sample space and events, probability- axioms of probability-some Elementary theorems- conditional probability.-Bayes Theorem.

UNIT-II (8 Lectures)

UNIT-III (12 Lectures)

INTERPOLATION: Finite differences: Forward, Backward and Central differences - Other difference operators and relations between them - Differences of a polynomial – Missing terms - Newton’s forward interpolation, Newton’s backward interpolation, Interpolation with unequal intervals – Lagrange’s interpolation.

UNIT-IV (8 Lectures)
Curve Fitting: Method of least squares - Fitting a straight line, second degree parabola and non-linear curves of the form by the method of least squares.

UNIT-V (8 Lectures)

TEXT BOOKS:

REFERENCE BOOKS:
UNIT-I

PROBABILITY
The study of probability provides a mathematical framework for assumptions made and is essential in every decision making. It is becoming prominent by its applications in many fields of learning, such as insurance, statistics, life sciences, and engineering.

The study of probability enables to focus a detailed study related to design of machines, undertake statistical quality control test and project control management.

Random Experiments:
An experiment whose outcome or result is not unique and therefore cannot be predicted with certainty.
E.g., In tossing a coin, head (H) or tail (T) may occur.
Throwing a die.

Sample Space:
The set of possible outcomes of a random experiment is called sample space. It is denoted by \( S \). The elements of sample space are called sample points.
E.g., In tossing a coin, \( S = \{ H, T \} \)
In rolling a die, \( S = \{ 1, 2, 3, 4, 5, 6 \} \)

Event:
Any subset of sample space is called an event.

Sure event:
Sample space \( (S) \) itself is called sure event or universal event or certain event.
The empty set (Ø) is called the impossible event.

* **Exhaustive event** -
  - A set of events is said to be exhaustive, if it includes all the possible events.

* **Mutually exclusive events** -
  - If the occurrence of one of the events precludes the occurrence of all other then such a set of events is said to be mutually exclusive.

* **Equally likely events** -
  - If one of the events cannot be expected to happen in preference to another then such events are said to be equally likely.

**Examples:**

1. When a coin is tossed, getting head (H) or tail (T) are two exhaustive, mutually exclusive and equally likely events.

2. When a die is thrown the turning up of the sixth different faces of the die are exhaustive, mutually exclusive and equally likely events.

* **Odds in favour of an event** -
  - If the no. of cases favourable to event (e) is m, and the no. of cases not favourable to e is n then odds in favour of e = \( \frac{m}{n} \)
* Classical definition of Probability: (chance)

If there are 'n' exhaustive, mutually exclusive and equally likely cases of which 'm' are favourable to an event E, then a probability of happening E, Pr(E) is defined as

\[ Pr(E) = \frac{\text{no. of cases favourable to E}}{\text{no. of exhaustive cases}} = \frac{m}{n} \]

* Axioms of Probability:

i) For any arbitrary event A of S, \( 0 \leq Pr(A) \leq 1 \)

ii) \( Pr(S) = 1 \) i.e., probability of sure event is 1.

iii) For any two mutually exclusive events A & B \( \cap S, \)

\[ Pr(A \cup B) = Pr(A) + Pr(B) \]

* Some Elementary Theorems:

1. \( Pr(\emptyset) = 0 \)

   let A be any arbitrary event of S

   we have \( A \cup \emptyset = A \)

   \[ \therefore Pr(A \cup \emptyset) = Pr(A) \]

   \[ \Rightarrow Pr(A) + Pr(\emptyset) = Pr(A) \] \[ \therefore A \cap \emptyset \text{ are mutually exclusive} \]

   \[ \Rightarrow Pr(\emptyset) = 0 \]

2. \( Pr(A^c) = 1 - Pr(A) \)

   let A be arbitrary event of S and \( A^c \) is event of non-occurrence of A.

   we have \( A \cup A^c = S \)

   \[ \therefore Pr(A \cup A^c) = Pr(S) \]

   \[ \therefore Pr(A) + Pr(A^c) = Pr(S) = 1 \] \[ \therefore \text{by axiom (ii) \& (iii)} \]

   \[ Pr(A^c) = 1 - Pr(A) \]
Since \( P(A) \geq 0 \)
\[- P(A) \leq 0\]
\[\therefore P(A^c) = 1 - P(A) \geq 1\]

3. If \( A, B \) are any two arbitrary events of \( S \) then
\[P(A^c \cap B) = P(B) - P(ANB)\]

From fig, we have
\[(ANB) \cup (A^c \cap B) = B\]
\[\therefore P[(ANB) \cup (A^c \cap B)] = P(B)\]
\[\Rightarrow P(ANB) + P(A^c \cap B) = P(B)\] [By axiom (iii), \((ANB) \cap (A^c \cap B)\) are mutually exclusive]
\[P(A^c \cap B) = P(B) - P(ANB)\]

Note: \( P(ANB^c) = P(A) - P(ANB)\)

4. Addition theorem of probability:

If \( A, B \) are any two arbitrary events of \( S \) then
\[P(A \cup B) = P(A) + P(B) - P(ANB)\]

Proof:
From fig, we have
\[A \cup B = A \cup (A^c \cap B)\]
\[P(A \cup B) = P[A \cup (A^c \cap B)]\]
\[= P(A) + P(A^c \cap B) \quad [:: A \cap (A^c \cap B) \text{ are mutually exclusive & by axiom (iii)}]\]
\[P(A \cup B) = P(A) + P(B) - P(ANB) \quad [:: \text{by theorem (iii)}]\]

5. If \( A, B, C \) are any three arbitrary events of \( S \) then
\[P(A \cup B \cup C) = P(A \cup (B \cup C))\]
\[= P(A) + P(B \cup C) - P(AN(B \cup C)) \quad (:: \text{by theorem 4})\]
7) A class consists of 6 girls and 10 boys. A committee of 5 students to be selected from the class. Find the probability for the committee to contain i) 4 boys ii) exactly 3 girls iii) at least 4 girls.

Total no. of students = 6 + 10 = 16

No. of ways of selecting 5 students out of 16 = 16C₅ = N(s)

i) Let E₁ be the event that a committee contains 4 boys.

Then \( n(E₁) = 10C₄ \times 6C₁ \)

\[ p(E₁) = \frac{10C₄ \times 6C₁}{16C₅} = \frac{15}{52} = 0.2884 \]

ii) Let E₂ be the event that a committee contains exactly 4 girls. Then \( n(E₂) = 6C₄ \times 10C₁ \)

\[ p(E₂) = \frac{6C₄ \times 10C₁}{16C₅} = \frac{675}{4,368} = 0.1542 \]

iii) Let E₃ be the event that a committee contains at least 4 girls. Then \( n(E₃) = 6C₄ \times 10C₁ + 6C₅ \times 10C₀ \)

\[ p(E₃) = \frac{n(E₃)}{N(s)} = \frac{6C₄ \times 10C₁ + 6C₅}{16C₅} = \frac{1}{28} = 0.0357 \]

8) Three bulbs are chosen at random from 12 bulbs of which 5 are defective. Find the probability that i) All are defective ii) Two are defective.

No. of ways of selecting 3 bulbs out of 12 bulbs = 12C₃ = N(s)

i) Let E₁ be the event that all are defective.

There are 5 defective bulbs & 7 non-defective bulbs.
No. of ways of selecting all defective bulbs.

Then \( n(e_1) = \binom{5}{3} = 10 \)

\[ \therefore \ P(e_1) = \frac{n(e_1)}{n(S)} = \frac{10}{80} = \frac{1}{8} = 0.045 \]

ii) Let \( e_2 \) be the event that 2 are defective.

Then \( n(e_2) = \binom{5}{2} \times 7c_1 \)

\[ \therefore \ P(e_2) = \frac{\binom{5}{2} \times 7c_1}{12c_3} = \frac{70}{220} = \frac{7}{22} = 0.3181 \]

9) A bag contains 40 tickets numbered 1, 2, 3, ..., 39, ... 40, of which four are drawn at random and arranged in ascending order \( (t_1 < t_2 < t_3 < t_4) \). Find the probability of \( t_3 \) being 25.

SOL: No. of ways of drawing 4 tickets out of 40 = \( 40c_4 = 91390 \)

If \( t_3 = 25 \) then the two tickets \( t_1 \) & \( t_2 \) must come out of 24 tickets numbered 1, 2, ..., 23, 24. The other ticket \( t_4 \) must come out of 15 tickets numbered 26, 27, ..., 40.

\[ \therefore \ \text{No. of ways that } t_3 \text{ being 25} = \frac{24c_2 \times 15c_1}{1} = 414 \]

\[ \therefore \ \text{Required probability} = \frac{24c_2 \times 15c_1}{40c_4} = \frac{414}{91390} = 0.00453 \]

6) 'A' has one shape in a lottery in which there is one prize and two blanks; 'B' has three shapes in which there are 3 prizes & 6 blanks in a lottery; compare the probability of A's success to that of B's success.

SOL: No. of ways that A can draw a ticket = \( 3c_1 \)

No. of ways that A can get a prize = \( 3c_1 - 2c_1 = 1 \)
The prob. of A's success = \( \frac{1}{2} \)

No. of ways that B can draw a ticket = \( 9C_3 = 84 \)

No. of ways that B can get a prize = \( 9C_3 - 6C_2 = 65 \)

\[ \therefore \text{The prob. of B's success} = \frac{16}{21} = 0.7619 \]

Hence A's probability of success : to B's probability of success = \( \frac{1}{2} : \frac{16}{21} = \frac{7}{16} \)

(i) A man's pocket contains 5 fifty paise coins, 4 '20' paise coins and 4 ten paise coins. A boy is asked to draw a coin at random. What is the probability of the boy drawing:

i) max possible amount

ii) min possible amount

iii) coins of diff. value?

Set: Total no. of coins = 5 + 4 + 4 = 13

No. of ways of drawing 2 coins out of 13 = \( 13C_2 = 78 \)

i) No. of ways of drawing max amount = 5C_2

\[ \therefore \text{Required probability} = \frac{5C_2}{13C_2} = \frac{5}{39} = 0.1282 \]

ii) No. of ways of drawing min amount = 4C_2

\[ \therefore \text{Required probability} = \frac{4C_2}{13C_2} = \frac{1}{13} = 0.0769 \]

iii) No. of ways of drawing diff. values = \( 5C_1 \times 4C_1 + 4C_1 \times 4C_1 \)

\[ \therefore \text{Required probability} = \frac{5 \times 4 + 4 \times 4}{18C_2} = \frac{56}{78} = \frac{28}{39} = 0.7179 \]
**Independent events**

Two events are said to be independent, if happening or

- failure of one does not affect the happening or failure of the other.

If A & B are any two independent events then

\[ P(AB) = P(A) \cdot P(B) \]

**Results:** If A, B are any two independent events

i) Then A & B\(^c\) are independent.

ii) A\(^c\) & B are independent.

iii) A\(^c\) & B\(^c\) are independent.

**Proof:** Since A & B are independent, \( P(AB) = P(A) \cdot P(B) \)

i) \[ P(A \cap B^c) = P(A) - P(AB) \]

\[ = P(A) - P(A) \cdot P(B) = P(A) \left[ 1 - P(B) \right] = P(A) \cdot P(B^c) \]

\[ \therefore A \ & B^c \ are \ independent. \]

ii) \[ P(A^c \cap B) = P(B) - P(AB) \]

\[ = P(B) - P(A) \cdot P(B) = P(B) \left[ 1 - P(A) \right] = P(B) \cdot P(A^c) \]

\[ \therefore A^c \ & B \ are \ independent. \]

iii) \[ P(A^c \cap B^c) = P[(A \cup B)^c] = 1 - P(A \cup B) \]

\[ = 1 - [P(A) + P(B) - P(AB)] = 1 - P(A) - P(B) + P(A) \cdot P(B) \]

\[ = 1 - P(A^c) + P(B) \cdot P(A) - P(B) \]

\[ = [1 - P(A)] - P(B) [1 - P(A)] = [P - P(A)] [1 - P(B)] \]

\[ = P(A^c) \cdot P(B^c) \]

\[ \therefore A^c \ & B^c \ are \ independent. \]
8. The probability that Susheel will solve a problem is \( \frac{1}{4} \) and the probability that Ram will solve it is \( \frac{2}{3} \). If Susheel and Ram work independently, what is the prob. that the problem will be solved by a) both of them, b) at least one of them.

So, prob. that Susheel will solve the problem \( = \frac{1}{4} = P(S) \)

Prob. that Ram will solve the problem \( = P(R) = \frac{2}{3} \)

a) \( P(\text{both solve}) = P(S \cap R) = P(S) \cdot P(R) \)
\[ = \frac{1}{4} \times \frac{2}{3} = \frac{1}{6} \]

b) \( P(\text{atleast one of them}) = 1 - P(\text{none solve}) \)
\[ = 1 - P(S^c \cap R^c) \times 1 - P(S^c) \cdot P(R^c) = 1 - \frac{3}{4} \times \frac{1}{3} \]
\[ = 1 - \frac{1}{4} = \frac{3}{4} \]

9. The prob. that a 50 year old man will be alive at 60 is 0.83 and the prob. that a 45 year old women will be alive at 55 is 0.87. What is the chance that a man who is 50 & his wife who is 45 will both be alive 10 years hence?

So, prob. that a 50 year old man will be alive at 60 \( = P(M) = 0.83 \)

Prob. that a 45 year old women will be alive at 55 \( = P(W) = 0.87 \)

\[ \therefore \text{ Req. prob} = P(M \cap W) \]
\[ = P(M) \cdot P(W) = (0.83)(0.87) = 0.7221 \]

10. Box 'A' contains 5 red & 3 white marbles. Box 'B' contains two red & 6 white marbles. If a marble is drawn from each box, what is the prob. that they are both of same colour. (i.e. both are either red or white.)

So, prob. that a marble is drawn from box 'A' and is red
\[ = \frac{1}{8} \times \frac{5}{8} = \frac{5}{64} \]
\[ P(\text{selecting a box}) \]

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Prob. that a marble is drawn from box B and is red

\[ P(B_R) = \frac{1}{2} \times \frac{3}{8} = \frac{3}{16} \]

\[ P(A_B) = \frac{1}{4} \times \frac{5}{8} = \frac{5}{32} \]

\[ P(B_B) = \frac{1}{4} \times \frac{2}{8} = \frac{1}{16} \]

\[ \therefore \text{Req. prob.} = P(A_R\|NB) + P(A_N\|NB_2) \]

\[ = P(A_R) \cdot P(B_R) + P(A_N) \cdot P(B_N) \]

\[ = \frac{5}{16} \times \frac{1}{8} + \frac{3}{16} \times \frac{3}{8} = \frac{7}{64} = 0.1094 \]

11. Two aeroplanes bombed a target in succession. The prob. of each correctly scoring a hit is 0.3 and 0.2 respectively. Find the prob. that

i) The target is hit

ii) both fails to score hit

iii) The target is hit by the second plane.

Let \( P(A_1) \), \( P(A_2) \) be the probabilities that the target is being hit by the two aeroplanes \( A_1 \), \( A_2 \) respectively.

\[ P(A_1) = 0.3 \quad P(A_2) = 0.2 \]

Prob. that \( A_1 \) fails to hit the target = \( P(A_1^c) = 0.7 \)

Prob. that \( A_2 \) fails to hit the target = \( P(A_2^c) = 0.8 \)

i) \( P(\text{target is hit}) = P(A_1) + P(A_1^c \cap A_2) \]

\[ = P(A_1) + P(A_1^c) \cdot P(A_2) \]

\[ = 0.3 + (0.7)(0.2) = 0.44 \]

ii) \( P(\text{both fails to score hit}) = P(A_1^c \cap A_2^c) \)

\[ = P(A_1^c) \cdot P(A_2^c) \]

\[ = 0.7 \times 0.8 = 0.56 \]

iii) \( P(\text{target is hit by } A_2) = P(A_1^c \cap A_2) \)

\[ = P(A_1^c) \cdot P(A_2) = (0.7)(0.2) = 0.14 \]
A pair of dice is tossed twice. Find the prob of scoring 7 points (getting sum 7), i) once ii) at least one iii) twice.

Let \( P(T_1), P(T_2) \) are the probabilities of a dice being tossed scoring 7 points in the first and second toss, respectively.

\[ P(T_1) = P(T_2) = \frac{6}{36} = \frac{1}{6} \]

\[ P(T_1^c) = P(T_2^c) = \frac{30}{36} = \frac{5}{6} \]

i) \( P(\text{scoring 7 points once}) = P(T_1 \cap T_2^c) + P(T_1^c \cap T_2) \)

\[ = P(T_1) \cdot P(T_2^c) + P(T_1^c) \cdot P(T_2) \]
\[ = \frac{1}{6} \times \frac{5}{6} + \frac{5}{6} \times \frac{1}{6} = \frac{5+5}{36} = \frac{10}{36} = \frac{5}{18} = 0.2777 \]

ii) \( P(\text{atleast once}) = P(T_1 \cap T_2^c) + P(T_1^c \cap T_2) + P(T_1 \cap T_2) \)

\[ = P(T_1) \cdot P(T_2^c) + P(T_1^c) \cdot P(T_2) + P(T_1) \cdot P(T_2) \]
\[ = \frac{1}{6} \times \frac{5}{6} + \frac{5}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{6} = \frac{5+5+1}{36} = \frac{11}{36} = 0.3055 \]

iii) \( P(\text{twice}) = P(T_1 \cap T_2) \)

\[ = P(T_1) \cdot P(T_2) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} = 0.0278 \]

13. If an avg of 1 birth in 80 is a case of twins, what is the prob that their will be at least one case of twins in a maternity hospital on a day when 80 births occur?

Let \( P(\text{twins will take birth in a day}) = \frac{1}{80} \)

\( P(\text{twins will not take birth in a day}) = \frac{79}{80} \)

\[ P(\text{twins will not take birth on a particular day when 80 births occur}) = \left( \frac{79}{80} \right)^{80} \]

\[ \therefore \text{Req. prob} = 1 - \left( \frac{79}{80} \right)^{80} \approx 0.8249 \]
Conditional Probability

If A and B are events in a sample space and \( P(A) \neq 0 \),

Then the prob. of B, after the event A has already occurred,
is called the conditional probability of B, given A and
is denoted by \( P(B|A) \) or \( P(B \mid A) \)

We define \( P(B|A) = \frac{P(AB)}{P(A)} \)

Thus \( P(A|B) = \frac{P(AB)}{P(B)} \), where \( P(B) \neq 0 \)

Multiplication Theorem of Probability

If A, B are any two events of a random experiment
such that \( P(A) \neq 0, P(B) \neq 0 \) then

i) \( P(AB) = P(A) \cdot P(B|A) \)

ii) \( P(\overline{A}B) = P(B) \cdot P(A|\overline{B}) \)

Eq(1): Let A and B be two events with \( P(A) = \frac{1}{4} \) and \( P(B) = \frac{1}{3} \),

\( P(AB) = \frac{1}{2} \).
Evaluate the following.

i) \( P(A|B) \)

ii) \( P(B|A) \)

iii) \( P(A|\overline{B}) \)

iv) \( P(\overline{A}|B) \)

v) \( P(\overline{A}|\overline{B}) \)

Sols:
By addition theorem, we have

\( P(AB) = P(A) + P(B) - P(AB) \)

\( = \frac{1}{4} + \frac{1}{3} - \frac{1}{2} = \frac{1}{12} \)

i) \( P(A|B) = \frac{P(AB)}{P(B)} = \frac{1/12}{1/3} = \frac{1}{4} \)

ii) \( P(B|A) = \frac{P(AB)}{P(A)} = \frac{1/12}{1/4} = \frac{1}{3} \)

iii) \( P(\overline{A}|B) = \frac{P(A^c \cap B)}{P(B)} = \frac{P(B) - P(AB)}{P(B)} = \frac{1}{3} - \frac{1}{12} = \frac{1}{4} \)

\( \frac{1}{3} \)

\( = \frac{8}{4} \)
**Baye's Theorem**

If \( e_1, e_2, \ldots, e_n \) are \( n \) mutually exclusive and exhaustive events such that \( P(e_i) \neq 0 \) for \( i = 1, 2, 3, \ldots, n \) and \( A \) is any arbitrary event with \( P(A) > 0 \), then

\[
P(e_i | A) = \frac{P(e_i \cap A)}{P(A)} = \frac{\sum_{i=1}^{n} P(e_i) \cdot P(A | e_i)}{P(A)}
\]

**Proof**

Let \( S \) be the sample space of a random experiment then

\[
S = e_1 \cup e_2 \cup e_3 \cup \ldots \cup e_n
\]

We have \( A = S \cap A = [e_1 \cup e_2 \cup \ldots \cup e_n] \cap A \)

\[
= (e_1 \cap A) \cup (e_2 \cap A) \cup \ldots \cup (e_n \cap A)
\]

\[
P(A) = P[(e_1 \cap A) \cup (e_2 \cap A) \cup \ldots \cup (e_n \cap A)]
\]

\[
P(A) = P(e_1 \cap A) + P(e_2 \cap A) + \ldots + P(e_n \cap A)
\]

\( (\because e_1 \cap A, e_2 \cap A, \ldots, e_n \cap A \) are mutually exclusive\)

\[
P(A) = P(e_1) \cdot P(A | e_1) + P(e_2) \cdot P(A | e_2) + \ldots + P(e_n) \cdot P(A | e_n)
\]

\[
P(A) = \sum_{i=1}^{n} P(e_i) \cdot P(A | e_i)
\]

[\( \because \) By Multiplication Theorem]

By the definition of conditional probability

\[
P(e_i | A) = \frac{P(e_i \cap A)}{P(A)} = \frac{P(e_i) \cdot P(A | e_i)}{P(A)}
\]

[By (i)]

Which is known as Baye's Theorem.
1. There are three bags: first containing 1 white, 2 red, 3 green balls, second a white, 3 red, 1 green balls and third 3 white, 1 red, 2 green balls. Two balls are drawn from a bag chosen at random. These are found to be one white and one red. Find the prob. that the balls so drawn came from the second bag.

Let \( B_1, B_2, B_3 \) be the events of selecting first bag, second bag and third bag respectively.

\[ P(B_1) = P(B_2) = P(B_3) = \frac{1}{3} \]

Let 'N' be the event of drawing two balls in which one is red and another is white.

\[ P(A/B_1) = \frac{1 \times 2 \times 3}{6} = \frac{3}{15} \]

\[ P(A/B_2) = \frac{3 \times 1 \times 3}{6} = \frac{6}{15} \]

By Baye's Theorem

\[ P(B_2/A) = \frac{P(B_2) \cdot P(A/B_2)}{P(B_1) \cdot P(A/B_1) + P(B_2) \cdot P(A/B_2) + P(B_3) \cdot P(A/B_3)} \]

\[ = \frac{\frac{1}{3} \times \frac{6}{15}}{\frac{1}{3} \left[ \frac{3}{15} + \frac{6}{15} + \frac{2}{15} \right]} = \frac{\frac{3 \times 2}{15}}{\frac{11}{15}} = \frac{6}{11} \]

2. In a certain college, 4% of the boys and 1% of the girls are taller than 1.8 m further more 60% of the students are girls. If a student is selected at random and is found to be taller than 1.8 m. What is the prob. that the student is a) boy ii) girl?

Let \( B \) be the event of selecting a boy and \( G \) be that of girl.

\[ P(B) = \frac{40}{100} = \frac{4}{10}, \quad P(G) = \frac{60}{100} = \frac{6}{10} \]
Let $A$ be the event of selecting a student who is taller than $1.8$ m. \[ p(A|B) = \frac{4}{100}, \quad p(A|C) = \frac{1}{100} \]

By Baye's Theorem

i) \[ p(B/A) = \frac{p(B) \cdot p(A|B)}{p(B) \cdot p(A|B) + p(C) \cdot p(A|C)} = \frac{\frac{4}{100} \times \frac{4}{10}}{\frac{4}{10} \times \frac{4}{100} + \frac{6}{10} \times \frac{1}{100}} = \frac{16}{16 + 64} = \frac{16}{80} = \frac{8}{40} = \frac{8}{11} \]

\[ p(E/A) = \frac{6}{10} \times \frac{1}{100} = \frac{6}{10} \times \frac{1}{100} + \frac{4}{10} \times \frac{4}{100} \]

\[ = \frac{6}{6 + 16} = \frac{6}{22} = \frac{3}{11} \]

8. In a bolt factory, machines $A$, $B$, and $C$ manufacture 25%, 85%, and 40% of the total. Of their output, $5\%$, $4\%$, and $2\%$ are defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the probabilities that it was manufactured by machines $A$, $B$, or $C$?

Let $E_1$, $E_2$, $E_3$ be the events that a bolt is manufactured by machines $A$, $B$, or $C$ respectively.

\[ p(M_1) = \frac{25}{100} = 0.25, \quad p(M_2) = \frac{35}{100} = 0.35, \quad p(M_3) = \frac{40}{100} = 0.40 \]

Let $D$ be the event of drawing a defective bolt.

\[ p(D/M_1) = \frac{5}{100} = 0.05, \quad p(D/M_2) = \frac{4}{100} = 0.04, \quad p(D/M_3) = \frac{2}{100} = 0.02 \]

By Baye's Theorem

\[ p(M_1/D) = \frac{p(M_1) \cdot p(D/M_1)}{p(M_1) \cdot p(D/M_1) + p(M_2) \cdot p(D/M_2) + p(M_3) \cdot p(D/M_3)} \]

\[ = \frac{(0.25)(0.05)}{(0.25)(0.05) + (0.35)(0.04) + (0.4)(0.02)} \]
\[ P(M_1|D) = \frac{0.0125}{0.0125 + 0.014 + 0.008} = \frac{2.5}{69} = 0.3623 \]

\[ P(M_2|D) = \frac{P(M_2) \cdot P(D|M_2)}{P(M_1) \cdot P(D|M_1) + P(M_2) \cdot P(D|M_2) + P(M_3) \cdot P(D|M_3)} \]

\[ = \frac{(0.04) (0.25)}{0.0125 + 0.014 + 0.008} = \frac{2.8}{69} = 0.040579 \]

\[ P(M_3|D) = \frac{P(M_3) \cdot P(D|M_3)}{P(M_1) \cdot P(D|M_1) + P(M_2) \cdot P(D|M_2) + P(M_3) \cdot P(D|M_3)} \]

\[ = \frac{(0.008)}{0.0125 + 0.014 + 0.008} = \frac{16}{69} = 0.23188 \]
UNIT-II

RANDOM VARIABLES & DISTRIBUTIONS
If \( f(x) \) satisfy the following conditions

1) \( f(x) \geq 0 \) for all \( x \)
2) \( \sum_{x} f(x) = 1 \)

then \( f(x) \) is called discrete probability function or probability mass function (p.m.f.)

**Cumulative distribution function (c.d.f.)**

The cumulative distributive function \( F(x) \) of a D.R.V. \( X \)

is \( F(x) = P(X \leq x) = \sum_{x \leq x} P(X = x) \)

for eg: \( F(2) = P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) \)

**Continuous Probability distribution:**

A continuous function \( f(x) \) of a continuous random variable \( x \) satisfying the following conditions

1) \( f(x) \geq 0 \) for all \( x \)
2) \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \)
   
   i.e., the area bounded by the curve \( y = f(x) \) and the axis is unity is called continuous probability function (or) probability density function (p.d.f.)

The cumulative distribution function \( F(x) \) of a continuous random variable \( x \) is defined by

\[
F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x) \, dx
\]

for eg: \( F(2) = P(X \leq 2) = \int_{-\infty}^{2} f(x) \, dx \)

**Properties of cumulative distribution function:**

1) \( 0 \leq F(x) \leq 1 \)
2) \( f(x) = \frac{d}{dx} \left\{ F(x) \right\} \) where \( f(x) \) is p.m.f / p.d.f
3) a) \( F(-\infty) = 0 \)  \( \) b) \( F(\infty) = 1 \)
\[ P(a \leq x \leq b) = \int_{a}^{b} f(x) \, dx = F(b) - F(a) \]

The mean or expectation of a random variable \( x \) is denoted by \( \mu \) or \( E(x) \) and the variance is denoted by \( \sigma^2 \) or \( V(x) \).

The positive square root of variance is called the standard deviation (S.D) and is denoted by \( \sigma \).

Formulae:

**Discrete**

i) \( \mu = E(x) = \sum_{i=1}^{n} x_i \cdot P(x=x_i) \)

ii) \( \sigma^2 = V(x) = \sum_{i=1}^{n} x_i^2 \cdot P(x=x_i) - \mu^2 \)

i.e. \( V(x) = E(x^2) - [E(x)]^2 \)

**Continuous**

i) \( \mu = E(x) = \int_{-\infty}^{\infty} x \cdot f(x) \, dx \)

ii) \( \sigma^2 = V(x) = \int_{-\infty}^{\infty} x^2 \cdot f(x) \, dx - \mu^2 \)

i.e. \( V(x) = E(x^2) - [E(x)]^2 \)

---

1. A random variable \( x \) has the following probability function:

<table>
<thead>
<tr>
<th>( x = x_i )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(x=x_i) )</td>
<td>( k )</td>
<td>( 3k )</td>
<td>( 5k )</td>
<td>( 7k )</td>
<td>( 9k )</td>
<td>( 11k )</td>
<td>( 13k )</td>
</tr>
</tbody>
</table>

i) Find \( k \)

ii) Mean

iii) Variance

iv) \( P(x \leq 4) \)

v) \( P(x \geq 5) \)

vi) \( P(3 \leq x \leq 6) \)

vii) What will be the minimum value of \( k \) so that \( P(x \leq 2) > 0.8 \)

Sol:-

i) \( \sum_{i=1}^{n} \frac{2}{(2k)^2} \cdot P(x=x_i) = 1 \)

i.e. \( k + 3k + 5k + 7k + 9k + 11k + 13k = 1 \)

\[ \Rightarrow k = \frac{1}{49} \]

ii) \( \text{Mean} = \mu = \sum_{i=1}^{n} x_i \cdot P(x=x_i) \)

\[ = (0 \times 1) + (1 \times 3k) + (2 \times 5k) + (3 \times 7k) + (4 \times 9k) + (5 \times 11k) + (6 \times 13k) \]
\[ \mu = \frac{203}{49} = \frac{29}{4} \]

iii) variance = \[ \sigma^2 = \sum_{i=1}^{2} \alpha_i^2 p(x=x_i) - \mu^2 \]

\[ = \left(0.8k\right) + \left(1.3k\right) + \left(4.5k\right) + \left(9.7k\right) + \left(16.9k\right) + \left(95.11k\right) + \left(36.13k\right) \]

\[ = 9.73K - \left(\frac{29}{4}\right)^2 = \frac{9.73}{9.9} - \left(\frac{29}{4}\right)^2 = \frac{132}{49} \]

iv) \[ P(x \leq 4) = P(x=0) + P(x=1) + P(x=2) + P(x=3) \]

\[ = K + 5k + 5k + 7k \]

\[ = 16K = \frac{16}{49} \]

v) \[ P(x \geq 5) = P(x=5) + P(x=6) \]

\[ = 11K + 13K \]

\[ = 24K = \frac{24}{49} \]

vi) \[ P(3 \leq x \leq 6) = P(x=4) + P(x=5) + P(x=6) \]

\[ = 9K + 11K + 13K \]

\[ = 33K = \frac{33}{49} \]

vii) \[ P(x \leq 2) = P(x=0) + P(x=1) + P(x=2) \]

\[ = K + 8K + 5K \]

\[ = 9K = \frac{9}{49} \]

Given \[ P(x \leq 2) \geq 0.8 \Rightarrow 9K \geq 0.8 \]

\[ \Rightarrow \frac{0.8}{9} = \frac{1}{60} \]

\[ K \geq \frac{1}{30} \]

\[ \therefore \] Thus min value of \( k = \frac{1}{30} \).
The probability mass function of a variable $X$ is given below:

<table>
<thead>
<tr>
<th>$x = x_i$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(x_i)$</td>
<td>0</td>
<td>$k$</td>
<td>$2k$</td>
<td>$2k$</td>
<td>$3k^2$</td>
<td>$2k^3$</td>
<td>$k^3 + k$</td>
</tr>
</tbody>
</table>

Determine:
1) $k$
2) $p(x = 6)$
3) $p(X > 6)$
4) $p(0 < x < 5)$
5) If $p(x = k) = \frac{1}{2}$, find the min value of $k$.

i) \[ \sum_{i=1}^{n} p(x = x_i) = 1 \]

\[ 0 + K + 2K + 3K + K^2 + 2K^2 + 2K^3 + K = 1 \]
\[ 10K^2 + 9K - 1 = 0 \]
\[ 10K^2 + 10K − K − 1 = 0 \]
\[ 10K(K + 1) − 1(K + 1) = 0 \]
\[ (10K − 1)(K + 1) = 0 \]
\[ K = \frac{1}{10} \text{ or } -1 \]
\[ \therefore K = \frac{1}{10} \]

iii) $p(x \geq 6) = p(x = 6) + p(x = 7)$

\[ = 8K^2 + 7K^2 + K \]
\[ = 9K^2 + K \]
\[ = \frac{9}{100} + \frac{1}{100} = \frac{9 + 10}{100} \]
\[ p(x \geq 6) = \frac{19}{100} \]

iv) For $k = \frac{1}{10}$

\[ p(x \leq 6) = p(x = 1) + p(x = 2) + p(x = 3) + p(x = 4) + p(x = 5) \]
\[ + p(x = 6) + p(x = 7) \]
\[ = 0 + K + 2K + 3K + 2K + K^2 \]
\[ = 8K + K^2 \]
\[ = \frac{81}{100} \]

v) $p(x \leq k) > \frac{1}{2}$

For $k = 1$:
\[ p(x \leq 1) = p(x = 0) + p(x = 1) = k = 0.1 < 0.5 \]

For $k = 2$:
\[ p(x \leq 2) = p(x = 0) + p(x = 1) + p(x = 2) = k + 2k = 3k = 0.3 = 0.5 \]
\[ k = 3 : p(x \leq 3) = p(x = 0) + p(x = 1) + p(x = 2) + p(x = 3) = 3k + 2k = 0.5 + 0.5 = 1 \]
\[ k = 4 : p(x \leq 4) = p(x = 0) + p(x = 1) + p(x = 2) + p(x = 3) + p(x = 4) = 3k + 2k = 0.8 > 0.5 \]

\[ \therefore \text{the min value of } k = 4. \]
A random variable $X$ has the following probability function:

$$P(X=x) = \begin{cases} \frac{3}{2} & x = -2, 0, 1 \\ \frac{1}{2} & x = 2, 3 \end{cases}$$

\[ \text{Find (i) } \mu \text{ (ii) } \sigma \text{ (iii) } \sigma^2 \]

i) \[ P(X=x) = \begin{cases} 0.1 & x = 0, 1, 2, 3 \end{cases} \]

\[ \mu = \sum_{x} x \cdot P(X=x) = -0.2 - 2 \cdot 0.2 + 2 \cdot 0.3 + 3 \cdot 0.1 = 0.8 \]

\[ \sigma^2 = \sum_{x} (x - \mu)^2 \cdot P(X=x) = \sum_{x} x^2 \cdot P(X=x) - \mu^2 = -0.8 - 4 \cdot 0.2 + 4 \cdot 0.3 + 9 \cdot 0.1 = 0.92 \]

\[ \sigma = \sqrt{\sigma^2} = \sqrt{0.92} \approx 0.96 \]

\[ \text{Find } \sigma \text{ (variance).} \]

\[ \text{Also find } \mu. \]

\[ f(x) = \begin{cases} kx & 0 \leq x < 2 \\ 2x - k & 2 \leq x < 4 \\ -kx + 10k & 4 \leq x \leq 6 \end{cases} \]

\[ \int_{-\infty}^{\infty} f(x) \, dx = 1 \]

\[ \int_{a}^{b} f(x) \, dx = \int_{0}^{2} kx \, dx + \int_{2}^{4} 2x - k \, dx + \int_{4}^{6} -kx + 10k \, dx \]

\[ = \frac{k}{2} x^2 \bigg|_{0}^{2} + 2x - k \bigg|_{2}^{4} - \frac{k}{2} x^2 \bigg|_{4}^{6} \]

\[ = 2k - 6k + 16 - 8k + 40 - 32k + 100k \]

\[ \Rightarrow 0 + k \left( \frac{32}{2} \right) + 2k(2k) + (6x - \frac{x^3}{3}) = 0 \]

\[ \Rightarrow 2k + 4k + k(12 - 16) = 1 \]

\[ \Rightarrow k = \frac{1}{8} \]

\[ \text{Find } \mu. \]

\[ \mu = \int_{-\infty}^{\infty} x \cdot f(x) \, dx = \int_{0}^{2} kx^2 \, dx + \int_{2}^{4} 2x^2 - kx \, dx + \int_{4}^{6} -kx^2 + 10kx \, dx \]

\[ = \frac{k}{3} x^3 \bigg|_{0}^{2} + \frac{2k}{3} x^3 \bigg|_{2}^{4} + \frac{10k}{3} x^3 \bigg|_{4}^{6} \]

\[ = \frac{k}{3} (8) + 2 \left( \frac{32}{3} \right) + 10 \left( \frac{216}{3} \right) \]

\[ \Rightarrow \frac{1}{8} \left( \frac{8}{3} + 12 + (108 - 32) \right) - \frac{64}{3} = 3 \]
Variance \( \sigma^2 = \int_{-\infty}^{\infty} x^2 f(x) \, dx - \mu^2 \)

\[
= \int_{0}^{2} x^2 f(x) \, dx + \int_{2}^{\infty} f(x) \, dx + \int_{0}^{\infty} \alpha f(x) \, dx - \mu^2 \\
= \int_{0}^{2} x^2 (kx) \, dx + \int_{2}^{\infty} x^2 (2x) \, dx + \int_{0}^{\infty} \alpha(x-\kappa) \, dx - \mu^2 \\
= k \left[ \left( \frac{a^3}{a} \right)^2 + 2k \left( \frac{a^3}{a} \right)^2 + 6k \left( \frac{a^3}{a} \right)^2 - k \left( \frac{a^3}{a} \right)^2 \right] - \mu^2 \\
= k \left[ \frac{16}{q} + 2k \left( \frac{8 - \frac{8}{3}}{3} \right) + 6k \left( \frac{2 \kappa - 2 \kappa}{3} \right) - k \left[ \frac{12q - 3q}{q} \right] \right] - \mu^2 \\
= k + \frac{102k}{3} + \frac{912}{3} k - \frac{300}{q} k - \mu^2 \\
= k \left[ \frac{38}{3} - q \right] - \mu^2 \\
= \frac{38}{3} - q = \frac{5}{3}
\]

6. A function is defined as follows:

\[
f(x) = \begin{cases} 
0, & 0 < x < 2 \\
\frac{1}{18} (2x+3), & 2 \leq x \leq q \\
0, & x > q, \ \text{or} \ \ x < 0
\end{cases}
\]

Show that it is a p.d.f.

Find the prob. that a variable \( x \) is in range:

- Clearly, \( f(x) \geq 0 \) + k

Also,

\[
\int_{-\infty}^{\infty} f(x) \, dx = \int_{0}^{2} f(x) \, dx + \int_{2}^{q} \frac{1}{18} (2x+3) \, dx + \int_{q}^{\infty} f(x) \, dx \\
= 0 + \frac{1}{18} (q^2 + 3q)^q + 0 \\
= \frac{1}{18} (2q - 10) = 1
\]

\( \therefore f(x) \) is a probability density function.

Req. prob. = \( P(2 \leq x \leq 3) \)

\[
= \int_{2}^{3} f(x) \, dx = \int_{2}^{3} \frac{1}{18} (2x+3) \, dx \\
= \frac{1}{18} (2x^2 + 3x)^3 \\
= \frac{4}{9}
\]
Is the function defined as follows a density function?

\[ f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \]

Clearly, \( f(x) = e^{-x} \) for \( x \geq 0 \) (i.e., range of \( x \) is \( \mathbb{R}^+ \)).

Also,

\[
\int_{-\infty}^{0} f(x) \, dx = \int_{0}^{\infty} e^{-x} \, dx = 0 - (e^{-x})_{0} = -0 - (-1) = 1.
\]

If so, determine the prob. that the variate having this density will fall in the interval \((1,2)\). Also find the cumulative probability function \( F(x) \).

:. Req. prob. = \( P(1 < x < 2) \)

\[
= \int_{1}^{2} f(x) \, dx = \int_{1}^{2} e^{-x} \, dx
\]

\[
= -(e^{-x})_{1} = e^{-1} - e^{-2} = 0.2836
\]

By the definition of c.d.f., we have

\[ F(x) = P(x \leq x) = \int_{-\infty}^{x} f(t) \, dt \]

:. \( F(2) = \int_{-\infty}^{2} f(x) \, dx = \int_{-\infty}^{0} f(x) \, dx + \int_{0}^{2} e^{-x} \, dx \)

\[
= -(e^{-x})_{0} = 1 - e^{-2} = 0.8697
\]

The probability density \( p(x) \) of a continuous random variable is given by \( p(x) = y_0 e^{-\alpha x} \), \(-\infty < x < \infty\) then prove that \( y_0 = \frac{1}{\alpha} \).

Find the mean and variance of the distribution.

Set: We have \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \)

i.e., \( \int_{-\infty}^{\infty} y_0 e^{-\alpha x} \, dx = 1 \)

\[
\Rightarrow \int_{-\infty}^{0} y_0 \cdot e^{-\alpha x} \, dx + \int_{0}^{\infty} y_0 \cdot e^{-\alpha x} \, dx = 1
\]

\[
\Rightarrow y_0 \left[ \int_{-\infty}^{0} e^{-\alpha x} \, dx + \int_{0}^{\infty} e^{-\alpha x} \, dx \right] = 1
\]

\[
\Rightarrow y_0 \left[ \frac{1}{-\alpha} e^{-\alpha x} \right]_{-\infty}^{0} + y_0 \left[ \frac{1}{-\alpha} e^{-\alpha x} \right]_{0}^{\infty} = 1
\]

\[
\Rightarrow y_0 \left[ 1 - \frac{e^{0} - e^{-\alpha} + e^{0}}{e^{0} - e^{-\alpha} + e^{0}} \right] = 1
\]

\[
\Rightarrow y_0 \left[ 1 - 0 - 0 + \frac{1}{1} \right] = 1 \Rightarrow 2y_0 = 1 \Rightarrow y_0 = \frac{1}{2}
\]
ii) \( \text{Mean (\mu)} = \int_{-\infty}^{\infty} \alpha \cdot f(\alpha) \cdot d\alpha \)

\[ = \int_{-\infty}^{\infty} \alpha \cdot y_0 \cdot e^{-\alpha} \cdot d\alpha \]

\[ = y_0 \left[ \left[ \int_{\infty}^{0} \alpha \cdot e^{-\alpha} \cdot d\alpha + \int_{0}^{\infty} \alpha \cdot e^{-\alpha} \cdot d\alpha \right] \right] \]

\[ = \frac{1}{\alpha} \left[ \left[ \int_{\infty}^{0} \alpha \cdot e^{-\alpha} \cdot d\alpha + \int_{0}^{\infty} \alpha \cdot e^{-\alpha} \cdot d\alpha \right] \right] \]

\[ = \frac{1}{\alpha} \left[ \left[ 1 - \int_{0}^{\infty} e^{-\alpha} \cdot \alpha \cdot d\alpha \right] \right] \]

\[ = \frac{1}{\alpha} \left[ (1) - (0) \right] = \frac{1}{\alpha} \left[ 1 + \phi \right] = 0 \]

iii) \( \varepsilon(\chi^2) = \int_{-\infty}^{\infty} \alpha^2 \cdot f(\alpha) \cdot d\alpha \)

\[ = \int_{-\infty}^{\infty} \alpha^2 \cdot y_0 \cdot e^{-\alpha} \cdot d\alpha \]

\[ = y_0 \left[ \left[ \int_{\infty}^{0} \alpha^2 \cdot e^{-\alpha} \cdot d\alpha + \int_{0}^{\infty} \alpha^2 \cdot e^{-\alpha} \cdot d\alpha \right] \right] \]

\[ = \frac{1}{\alpha} \left[ \left[ \int_{\infty}^{0} \alpha^2 \cdot e^{-\alpha} \cdot d\alpha + \int_{0}^{\infty} \alpha^2 \cdot e^{-\alpha} \cdot d\alpha \right] \right] \]

\[ = \frac{1}{\alpha} \left[ \left[ 1 - \int_{0}^{\infty} \alpha^2 \cdot e^{-\alpha} \cdot d\alpha \right] \right] \]

\[ = \frac{1}{\alpha} \left[ (1) - (0) \right] = \frac{1}{\alpha} \left[ 1 + \phi \right] = 0 \]

\[ \int e^{\alpha \cdot f(\alpha)} \cdot d\alpha = \frac{e^{\alpha x}}{\alpha} \left[ f(x) - \frac{f'(x)}{\alpha} + \frac{f''(x)}{\alpha^2} \cdot \frac{\alpha^2}{\alpha^2} + \cdots \right] \]

\[ = \frac{1}{\alpha} \left[ \left[ f(2 + 0) \right] - \left[ f(0 + 2) \right] \right] = \frac{1}{\alpha} (2 + 2) = \lambda \]

\[ \text{Variance} = e(\chi^2) - \mu^2 \]

\[ = \lambda - 0 = \lambda / \lambda \]
In this chapter, we discuss about the following three prob. distributions:

1. Binomial distribution (B.D) 
2. Poisson distribution (P.D) 

1. Binomial distribution

Consider a series of 'n' independent trials in which 'p' is the prob. of success & 'q' is the prob. of failure so that $p + q = 1$. The prob. of 'r' successes in 'n' trials is given by:

$$\binom{n}{r} p^r q^{n-r}$$

where $r = 0, 1, 2, \ldots, n$

The binomial distribution of a discrete random variable 'x' with parameter $n, p$ is defined by

$$P(x = r) = \binom{n}{r} p^r q^{n-r}, \quad r = 0, 1, \ldots, n$$

Note:

(i) $P(x = r) = b(x; n, p)$ where $q = 1 - p$

(ii) $\sum_{x=0}^{n} P(x = r) = \sum_{x=0}^{n} \binom{n}{r} p^r q^{n-r}$

$$= \binom{n}{0} p^0 q^n + \binom{n}{1} p^1 q^{n-1} + \cdots + \binom{n}{n} p^n q^0$$

$$= (p+q)^n = 1$$

2. Mean & Variance of a B.D:

Mean ($\mu$) = $\sum_{x=0}^{n} x \cdot P(x = r)$

$$= 0 \sum_{x=1}^{n} x \cdot P(x = r)$$

$$= \sum_{x=1}^{n} x \cdot \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=1}^{n} \frac{x \cdot n!}{x! (n-x)!} p^x q^{n-x}$$

$$= \frac{\sum_{x=1}^{n} x \cdot n(n-1) \cdots (n-x+1)}{n!(n-x)!} p^x q^{n-x}$$

$$= \frac{n(n-1) \cdots (n-x+1)}{x!(n-x)!} p^x q^{n-x}$$

$$(x-1)!$$

$$(n-x)!$$
\[ n \sum_{y=0}^{(n-1)} \frac{(n-1)!}{y![(n-(y+1))!]} p^{y+1} q^{n-(y+1)} = n \sum_{y=0}^{(n-1)} \frac{(n-1)!}{y![(n-(y+1))!]} p y \cdot q^{n-1-y} \]

\[ = n \sum_{y=0}^{(n-1)} (n-1)_c y \cdot p y \cdot q^{n-1-y} \]

\[ = n \sum_{y=0}^{(n-1)} (n-1)_c y \cdot p y \cdot q^{n-1-y} = n \sum_{y=0}^{(n-1)} (n-1)_c y \cdot p y \cdot q^{n-1-y} \]

\[ \therefore q + p = 1 \]

\[ \Rightarrow (\sum_{y=0}^{(n-1)} (n-1)_c y \cdot p y \cdot q^{n-1-y}) = n \sum_{y=0}^{(n-1)} (n-1)_c y \cdot p y \cdot q^{n-1-y} \]

\[ \Rightarrow \mu = n \rho \]

\[ E(x^2) = \sum_{x=0}^{\infty} \alpha^2 \cdot p(x=x) \]

\[ = \sum_{x=0}^{\infty} (\alpha^2 - x + x) f(x=x) \]

\[ = \sum_{x=0}^{\infty} (\alpha^2 - x + x) p(x=x) \]

\[ = \sum_{x=0}^{\infty} \alpha^2 p(x=x) + \sum_{x=0}^{\infty} \alpha \cdot p(x=x) \]

\[ = \sum_{x=0}^{\infty} \alpha^2 p(x=x) + \mu \quad \text{(by (ii))} \]

\[ = \sum_{x=0}^{\infty} \alpha^2 p(x=x) + \mu \frac{n \cdot p^2 \cdot q^{n-x} - n p}{\alpha (n-2)!} \quad \text{(by (i))} \]

\[ = \sum_{x=0}^{\infty} \alpha^2 p(x=x) + \mu \frac{n \cdot p^2 \cdot q^{n-x} + n p}{\alpha (n-2)!} \]

\[ = n(n-1) \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)! \cdot (n-x)!} \cdot p^2 \cdot q^{n-x} + np \]

\[ \quad \text{(put } (x-2) = y, \text{ so that } y+2 = x) \]

\[ = n(n-1) \sum_{y=0}^{(n-1)} \frac{(n-2)!}{y![(n-(y+2))!]} \cdot p^{y+2} \cdot q^{(n-(y+2))} + np \]

\[ = n(n-1) \sum_{y=0}^{(n-1)} \frac{(n-2)!}{y![(n-(y+2))!]} \cdot p^y \cdot q^{(n-1)-y} + np \]

\[ = n(n-1) \sum_{y=0}^{(n-1)} \frac{(n-2)!}{y![(n-(y+2))!]} \cdot p^y \cdot q^{(n-1)-y} + np \]

\[ = n(n-1) \sum_{y=0}^{(n-1)} (n-1)_c y \cdot p^y \cdot q^{(n-1)-y} + np \]

\[ \therefore q + p = 1 \]
\[ n(p-1)p^0(q+1)p^{n-2} + np \]
\[ = n(p-1)p^2 + np \]
\[ : \text{variance} \sigma^2 = e(x^2) - \mu^2 \]
\[ = n(p-1)p^2 + np - (np)^2 \]
\[ = np^2 - np^2 + np - np^2 = np(1-p) \]
\[ \therefore \sigma^2 = npq \quad (\because p+q = 1 \Rightarrow q = 1-p) \]

Eq 6 The mean \( \mu \) and variance \( \sigma^2 \) of the binomial distribution are \( 4 \) and \( 4/3 \) respectively. Find the B.D. \( p(x \geq 1) \)

Eq z Given mean \( \mu = np = 4 \) and variance \( \sigma^2 = npq = 4/3 \)

\[ \therefore \frac{npq}{np} = \frac{4/3}{4} \Rightarrow \frac{1}{3} = q \quad \therefore p = \frac{2}{3} \]

But \( np = 4 \Rightarrow n = 4 \times \frac{3}{2} = 6 \)

\[ \therefore \text{The B.D. is given by } \begin{align*}
p(x=x) &= \binom{n}{x} \left( \frac{2}{3} \right)^x \left( \frac{1}{3} \right)^{n-x}, \quad x=0,1,2, ..., 6 
\end{align*} \]

\[ \therefore \text{req. prob. } = p(x \geq 1) = 1 - p(x=0) = 1 - p(x=0) \]
\[ = 1 - \binom{6}{0} \left( \frac{2}{3} \right)^0 \left( \frac{1}{3} \right)^6 \]
\[ = 1 - \left( \frac{1}{3} \right)^6 = 0.9986 \]

2) Determine the B.D. for which mean \( \mu = 8 \) and \( \sigma^2 = \sigma^2 \) variance \( = 3 \). Also find \( p(x \leq 3) \).

Eq z Given \( np = 2 \) \( \sigma^2 \Rightarrow q = \frac{1}{2} \), \( p = \frac{1}{2} \)

\[ np + npq = 3 \]
\[ n \left( \frac{1}{2} + \frac{1}{4} \right) = 3 \Rightarrow n = 4 \]

\[ \therefore \text{The B.D. is given by } \begin{align*}
p(x=x) &= \binom{4}{x} \left( \frac{1}{2} \right)^x \left( \frac{1}{2} \right)^{4-x}, \quad x=0,1,2,3,4 \]
\end{align*} \]

\[ \therefore \text{req. prob. } = p(x \leq 3) = p(x=0) + p(x=1) + p(x=2) + p(x=3) \]
\[ = \binom{4}{0} \left( \frac{1}{2} \right)^4 + 4 \binom{4}{1} \left( \frac{1}{2} \right)^3 \left( \frac{1}{2} \right) + 4 \binom{4}{2} \left( \frac{1}{2} \right)^2 \left( \frac{1}{2} \right) + 4 \binom{4}{3} \left( \frac{1}{2} \right) \left( \frac{1}{2} \right)^3 \]
\[ = \left( \frac{1}{2} \right)^4 \left[ 1 + 4 + 6 + 4 \right] = \frac{15}{16} = 0.9375 \]
500 articles were selected at a random out of a batch containing 10,000 articles. So were found to be defective. How many defective articles would you reasonably expect to have in the whole batch?

SOL:- \[ n = 500 \]

\[ \text{Prob. of a defective article} = \frac{30}{500} = \frac{3}{50} = 0.06 \]

\[ \Rightarrow \text{No. of defective articles in the whole batch} = 10000 \times 0.06 = 600 \]

(2) 10 coins are tossed simultaneously. Find the prob. of getting
   a) exactly 7 heads  b) at least 7 heads  c) no heads.

SOL: Here \[ n = 10 \] \( x \) denotes the no. of heads

\[ \text{prob. of getting heads} = \frac{1}{2} = P \] \[ \Rightarrow q = \frac{1}{2} \]

i) \[ P(x=4) = \binom{10}{4} \left( \frac{1}{2} \right)^4 \left( \frac{1}{2} \right)^{10-4} \]
   \[ = 210 \times \left( \frac{1}{2} \right)^6 = 0.2051 \]

ii) \[ P(x \geq 7) = P(x=7) + P(x=8) + P(x=9) + P(x=10) \]
   \[ = 10 \binom{10}{7} \left( \frac{1}{2} \right)^{10} + 10 \binom{10}{8} \left( \frac{1}{2} \right)^{10} + 10 \binom{10}{9} \left( \frac{1}{2} \right)^{10} + 1 \binom{10}{10} \left( \frac{1}{2} \right)^{10} \]
   \[ = 0.1319 \]

iii) \[ P(x=0) = 10 \binom{10}{0} \left( \frac{1}{2} \right)^{10} = 0.0001 \]

(5) Determine the prob. of getting sum 6 exactly 3 times in 7 throws with a pair of fair dice.

SOL: Here \( n(s) = 6^2 = 36 \), \( n = 7 \)

let \( x \) denotes the no. of throws.

\[ \text{Prob. of getting the sum 6 in a single throw of pair of dice} \]
\[ \left(1,6\right), \left(2,4\right), \left(3,3\right), \left(4,2\right), \left(5,1\right) \]
\[ = \frac{5}{36} = P \]

\[ \Rightarrow \text{req. prob} = P(x=3) = \binom{7}{3} \left( \frac{5}{36} \right)^3 \left( \frac{31}{36} \right)^4 = 0.0516 \]
5. Suppose that on the average one person in 1000 marks a numerical entry in preparing income tax return (ITR). If 10,000 forms are selected at random and examined, find the prob. that 9, 9 or 8 of the forms will be in error.

Set: Here \( n = 10,000 \) and \( p = \frac{1}{1000} \) so that \( \lambda = np = 10 \)

\[ \therefore \text{Req. prob.} = P(x=6, 9 \text{ or } 8) \]

\[ = P(x=6) + P(x=7) + P(x=8) \]

\[ = e^{-\lambda} \frac{\lambda^6}{6!} + e^{-\lambda} \frac{\lambda^7}{7!} + e^{-\lambda} \frac{\lambda^8}{8!} \]

\[ = e^{-10} \left( \frac{10^6}{6!} + \frac{10^7}{7!} + \frac{10^8}{8!} \right) = 0.2657 \]

***(3) A distributor of bean seeds determines from extensive test that 5% of a large batch of seeds will not germinate. He sells the seeds in packets of 200 and guarantees 90% germination. Determine the prob. that a particular packet will violate the guarantee.

Set: Here \( n = 200.\)

The prob. of a non-germinating seed = \( p = \frac{5}{100} = 0.05 \) \( \therefore \lambda = 0.95 \)

\( \therefore \lambda = np = 200 \times 0.05 = 10. \)

A packet will violate guarantee if it contains more than 20 non-germinating seeds.

\[ \therefore \text{Req. prob.} = P(x \geq 20) \]

\[ = 1 - P(x \leq 20) \]

\[ = 1 - \sum_{x=0}^{20} \frac{e^{-10} 10^x}{x!} = 0.0016 \]

4. In a certain factory turning out laser blades, there is a small chance of 0.002 that any blade to be defective. The blades are supplied in packets of 40; use P.D. to calculate the approximate no. of packets containing no defective, 1 defective, 2 defective blades respectively in a consignment of 10,000 packets.

Set: Here \( n = 10,000.\)

Prob. of a defective blade = \( p = 0.002 \)

\( \therefore \lambda = np = 10 \times 0.002 = 0.02 \)
i) \( P(\text{no defective blade}) = P(x=0) \)
\[
= \frac{e^{-\lambda} \lambda^0}{0!} = e^{-0.02} = 0.9802
\]

\[ \therefore \text{No. of packets containing no. defective blade in a consignment of 10,000 packets} = 10,000 \times 0.9802 \\
= 9802 \]

ii) \( P(1 \text{ defective blade}) = P(x=1) \)
\[
= e^{-\lambda} \frac{\lambda^1}{1!} = e^{-0.02} \frac{(0.02)}{1} = 0.0196
\]

\[ \therefore \text{No. of packets containing 1 defective blade in consignment of 10,000 packets} = 10,000 \times 0.0196 = 196 \]

iii) \( P(2 \text{ defective}) = P(x=2) \)
\[
= e^{-\lambda} \frac{\lambda^2}{2!} = e^{-0.02} \frac{(0.02)^2}{2} = 0.002
\]

\[ \therefore \text{No. of packets containing two defective blade in consignment of 10,000 packets} = 10,000 \times 0.002 = 20 \]

\* Fitting P.D.:

ii) The frequency of accidents for shift A in a factory is as shown in the following table.

<table>
<thead>
<tr>
<th>Accidents per shift</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>180</td>
<td>92</td>
<td>24</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

Calculate the mean no. of the accidents per shift and compare with actual accidents.

\[ \therefore \text{Here N = 4 and } N = \Sigma f = 300 \]
\[ \therefore \text{Mean no. of accidents per shift} = \lambda = \frac{\Sigma f \cdot \lambda}{N} \]
\[ = 0.05 \]
The theoretical frequencies of a p.d are given by

\[-f(1) = \frac{300 \times e^{-0.5} \times (0.5)^0}{0!} = 180.1487 \approx 180\]

\[-f(1) = \frac{300 \times e^{-0.5} \times (0.5)^1}{1!} = 91.8736 \approx 92\]

\[-f(1) = \frac{300 \times e^{-0.5} \times (0.5)^2}{2!} = 23.4883 \approx 23\]

\[-f(2) = \frac{300 \times e^{-0.5} \times (0.5)^3}{3!} = 8.9828 \approx 4\]

\[-f(3) = \frac{300 \times e^{-0.5} \times (0.5)^4}{4!} = 0.5028\]

Accidents per shift:

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>122</td>
<td>60</td>
<td>15</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

\[\text{Actual frequency:} \quad 122 \quad 60 \quad 15 \quad 2 \quad 1\]

\[\text{Theoretical frequency:} \quad 180 \quad 92 \quad 23 \quad 4 \quad 1\]

Put a p.d to the following set of observations.

\[x : 0 \quad 1 \quad 2 \quad 3 \quad 4\]

\[f : 122 \quad 60 \quad 15 \quad 2 \quad 1\]

So, \(N = 4\) and \(N = \sum f = 300\)

\[\text{Mean, } \mu = \bar{x} = \frac{\sum f \cdot x}{N} = \frac{107}{200} = 0.5\]

The theoretical frequencies of p.d are given by

\[-f(x) = \frac{300 \times e^{-0.5} \times (0.5)^x}{x!}, \quad x = 0, 1, 2, 3, 4\]

\[-f(0) = \frac{300 \times e^{-0.5} \times (0.5)^0}{0!} = 121.83\]

\[-f(1) = \frac{300 \times e^{-0.5} \times (0.5)^1}{1!} = 60.65\]

\[-f(2) = \frac{300 \times e^{-0.5} \times (0.5)^2}{2!} = 30.22\]

\[-f(3) = \frac{300 \times e^{-0.5} \times (0.5)^3}{3!} = 15.0163\]

\[-f(4) = \frac{300 \times e^{-0.5} \times (0.5)^4}{4!} = 7.058\]

\[x = 0 \quad 1 \quad 2 \quad 3 \quad 4\]

\[\text{Actual frequency:} \quad 122 \quad 60 \quad 15 \quad 2 \quad 1\]

\[\text{Theoretical frequency:} \quad 180 \quad 92 \quad 23 \quad 4 \quad 1\]
3. Normal Distribution (N.D.):

Let us define a variable \( z = \frac{x - \mu}{\sigma} \) where \( x \) is a binomial variable with mean \( \mu (= np) \) & S.D. \( (\sigma) = npq \) so that \( z \) is a variable with mean zero and variance unity. In the limit as \( n \to \infty \), the distribution \( q \) becomes a continuous distribution extending from \(-\infty\) to \( \infty \).

Normal distribution is a continuous probability distribution \( q \) is defined as the limiting form of the binomial distribution for large value of \( n \). When neither \( p \) nor \( q \) is very small.

It is given by

\[
\phi(x) = N(\mu, \sigma) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2}
\]

Properties of N.D.:

1. The normal curve \( y = f(x) \) is smooth, regular, bell-shaped and symmetric about the y-axis.
2. The mean, median, mode of the normal distribution coincide.
3. The area under the normal curve \( y = f(x) \) above the x-axis from \(-\infty\) to \( \infty \) is unity.

Note:

1. If \( a < z_1 < z_2 \) then \( P(z_1 \leq z \leq z_2) = P(0 \leq z \leq z_2) - P(0 \leq z \leq z_1) \)
2. \( P(z \leq z_1) = P(-\infty < z \leq z_1) \)
   \[ = P(-\infty < z \leq 0) + P(0 \leq z \leq z_1) \text{, } z_1 \text{ is positive} \]
   \[ = 0.5 + P(0 \leq z \leq z_1) \]
$P(Z > Z_1) = P(Z_1 \leq Z < \infty) = P(0 \leq Z < \infty) - P(0 \leq Z \leq Z_1) = 0.5 - P(0 \leq Z \leq Z_1)$

4. $P(-Z_1 \leq Z \leq Z_3) = P(-Z_1 \leq Z \leq 0) + P(0 \leq Z \leq Z_3)$
   \[ = P(0 \leq Z \leq Z_1) + P(0 \leq Z \leq Z_3) \]

5. $P(-Z_1 \leq Z \leq Z_1) = P(-Z_1 \leq Z \leq 0) + P(0 \leq Z \leq Z_1) = P(0 \leq Z \leq Z_1) + P(0 \leq Z \leq Z_1)$
   \[ = 2 P(0 \leq Z \leq Z_1) \]

Example 1

1. If $X$ is a normal variable with mean 30 and $\sigma = 5$. Find the prob. that
   i) $26 \leq x \leq 40$
   ii) $x > 45$
   iii) $|x - 60| > 5$

   So, given $\mu = 30$ and $\sigma = 15$

   i) when $\mu_1 = 26$, $z_1 = \frac{x_1 - \mu}{\sigma} = \frac{26 - 30}{5} = -0.8$

   when $\mu_2 = 40$, $z_2 = \frac{x_2 - \mu}{\sigma} = \frac{40 - 30}{5} = 2$.

   \[ P(26 \leq x \leq 40) = P(-0.8 \leq z \leq 2) \]
   \[ = P(-0.8 \leq z \leq 0) + P(0 \leq z \leq 2) \]
   \[ = P(0 \leq z \leq 0.8) + P(0 \leq z \leq 2) = 0.2880 + 0.4772 = 0.7653 \]

   ii) Given, when $\mu = 45$, $z = \frac{x - \mu}{\sigma} = 3$

   \[ P(x \geq 45) = P(z \geq 3) = P(3 \leq z < \infty) \]
   \[ = P(0 \leq z < \infty) - P(0 \leq z \leq 3) \]
   \[ = 0.5 - 0.9987 = 0.0013. \]

***

iii) $P(|X - 30| > 5) = 1 - P(|X - 30| \leq 5)$
   \[ = 1 - P(-5 \leq X - 30 \leq 5) \]
   \[ = 1 - P(25 \leq X \leq 35) \]
   \[ = 1 - P(\frac{25 - 30}{5} \leq Z \leq \frac{35 - 30}{5}) \]
   \[ = 1 - P(-1 \leq Z \leq 1) \]
   \[ = 1 - 0.6826 = 1 - 0.6826 = 0.2174. \]
For a N.O variable with mean $\mu = 1$ and S.D is 3. Find the prob that

i) $3.43 \leq x \leq 6.19$

ii) $-1.43 \leq x \leq 6.19$

So if $\mu = 1$ and $\sigma = 3$

i) $P(3.43 \leq x \leq 6.19) = P\left(\frac{3.43-1}{3} \leq z \leq \frac{6.19-1}{3}\right) = P(0.81 \leq z \leq 1.73)$

$= P(0 \leq z \leq 1.73) - P(0 \leq z \leq 0.81)$

$= 0.9580 - 0.2910 = 0.6670$

ii) $P(-1.43 \leq x \leq 6.19) = P\left(\frac{-1.43-1}{3} \leq z \leq \frac{6.19-1}{3}\right) = P(-0.81 \leq z \leq 1.73)$

$= P(-0.81 \leq z \leq 0) + P(0 \leq z \leq 1.73)$

$= 0.4582 + 0.2910$

$= 0.7492$

The mean height of 500 students is 151 cm and the S.D is 15 cm.
Assuming that the heights are normally independent distributed, find how many students heights lie between 120 & 155 cm.

So if $\mu = 151$ cm & $\sigma = 15 = 0$

To find $P(120 \leq x \leq 155)$

when $x_1 = 120 \Rightarrow Z_1 = \frac{x_1 - \mu}{\sigma} = \frac{120-151}{15} = -2.07$

when $x_2 = 155 \Rightarrow Z_2 = \frac{x_2 - \mu}{\sigma} = \frac{155-151}{15} = 0.27$

$\therefore P(120 \leq x \leq 155) = P(-2.07 \leq z \leq 0.27)$

$= P(-2.07 \leq z \leq 0) + P(0 \leq z \leq 0.27)$

$= P(0 \leq z \leq 2.07) + P(0 \leq z \leq 0.27)$

$= 0.4805 + 0.1064 = 0.5869$

\[ \text{No. of students whose heights lie b/w 120 & 155 cm} \]

$= 500 \times P(120 \leq x \leq 155)$

$= 500 \times 0.5869$

$= 293.5$

$\approx 294$
15. If $x$ is normally distributed with mean $\mu$ and variance $\sigma^2$, then find $P(1x-\mu \geq 0.01)$?

Given $\mu=8$, $\sigma=0.1$

\[ P(1x-\mu \geq 0.01) = P(-0.01 \leq x - \mu \leq 0.01) \]
\[ = P\left(\frac{1.99 - \mu}{\sigma} \leq z \leq \frac{1.01 - \mu}{\sigma}\right) \]
\[ = P\left(\frac{1.99 - 8}{0.1} \leq z \leq \frac{1.01 - 8}{0.1}\right) = P(-0.14 \leq z \leq 0.14) \]
\[ = 2 \times P(0 \leq z \leq 0.1) = 2 \times 0.0398 = 0.0796 \]

Thus, $P(1x-\mu \geq 0.01) = 1 - P(1x-\mu \leq 0.01)$
\[ = 1 - 0.0796 = 0.9204 \]

**Normal Approximation to B.D.**

The N.D. can be used to approximate the B.D. Suppose the number of successes 'x' ranges from $x_1$ to $x_2$. For large n, the calculation of binomial probabilities is very difficult. In such cases, the binomial curve can be replaced by the normal curve and the required probability is computed.

For any success 'x', real class interval is $(x_1 - \frac{1}{2}, x_2 + \frac{1}{2})$ then $x_1$ must correspond to the lower limit of the $x_1$ class and $x_2$ to the upper limit of the $x_2$ class. Hence $x_1 = \left(\frac{x_1 - 1}{2}\right) - np$

and $x_2 = \left(\frac{x_2 + 1}{2}\right) - np$

\[ \frac{\sqrt{npq}}{\sqrt{npq}} \]

\[ \therefore \text{The reqd prob} = P(x_1 \leq z \leq x_2) \]
Find the prob. that out of 100 patients i) 84 to 89 incl. give, ii) fewer than 86 will survive a heart operation, given that the chance of survival is 0.9.

Here n = 100, p = 0.9 so that q = 0.1

⇒ np = 90 & \sqrt{npq} = \sqrt{9} = 3

i) when \( n_1 = 84 \), \( z_1 = \frac{(84 - 1/2) - 90}{\sqrt{90}} \)

\[ z_1 = \frac{(84 - 1/2) - 90}{3} = -3.166 = -2.17 \]

when \( n_2 = 95 \) ⇒ \( z_2 = \frac{(95 + 1/2) - 90}{\sqrt{90}} \)

\[ z_2 = \frac{(95 + 1/2) - 90}{3} = 1.83 \]

⇒ req prob = \( P(-2.17 \leq z \leq 1.83) \)

\[ = P(-3.17 \leq z \leq 0) + P(0 \leq z \leq 1.83) \]

\[ = 0.4850 + 0.4664 = 0.9514 \]

ii) when \( n_1 = 86 \), \( z_1 = \frac{(86 - 1/2) - 90}{\sqrt{90}} \)

\[ z_1 = \frac{(86 - 1/2) - 90}{3} = -1.5 \]

⇒ req prob = \( P(z \leq 1.5) \)

\[ = P(-\infty < z < 1.5) \]

\[ = 0.5 - P(1.5 < z < \infty) \]

\[ = 0.5 - 0.4332 \]

\[ = 0.0668 \]
UNIT-III

ALGEBRAIC AND TRANSCENDENTAL EQUATIONS, INTERPOLATION
CHAPTER-1: ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

An equation of the form \( a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0 \)
where \( a_0, a_1, \ldots, a_n \) are real nos is called algebraic eq of degree \( n \). Eg. \( x^3 + 11x + 5 = 0 \)

Transcendental equations are non-algebraic equations involving transcendental functions such as exponential, logarithmic and trigonometric functions.

Eg: \( x e^x = 1, x \log x - 2 = 0, x^e = \cos x \)

A general form of algebraic or transcendental eq is \( f(x) = 0 \) where \( f(x) \) is defined and continuous on \([a,b]\)

Any value \( x \) for which \( f(x) = 0 \) is known as the root or solution of \( f(x) = 0 \). Geometrically, the root \( q \) \( f(x) = 0 \) is the point where the curve \( y = f(x) \) meets the \( x \)-axis.

An algebraic eq of degree \( n \) has exactly \( n \) roots, real or complex where as the no of roots of transcendental equations are not known at all.

Examples:
1. \( x^3 - 4x - 9 = 0 \) has 3 roots, one real \& 2 complex.
2. \( \sin x = 2 \) has no real roots.
3. \( \sin x = \frac{1}{2} \) has 2 definite no of roots.
4. \( \sin x = \frac{x}{2} \) has 2 real roots.

To locate the root of \( f(x) = 0 \), we use the following theorem in calculus.

Intermediate value theorem:

If \( f(x) \) is continuous in \([a,b]\) and if \( f(a) \) \( f(b) \) are of opp sign then the eq \( f(x) = 0 \) has atleast one root between \( a \) \& \( b \).

Note: The root of \( f(x) = 0 \) will be unique in the interval \((a,b)\) if

either \( f(x) < 0 \) (or) \( f(x) > 0 \) \( x \in (a,b) \)

In this chapter we find the approximate roots of algebraic \& transcendental equations using the following numerical methods.
1) Bisection method (bolzano method):-
Consider an equation \( f(x) = 0 \) in \([a, b]\).

Choose \( a, b \) such that \(-f(a), f(b) < 0\). Then a root \( q \) of \( f(x) = 0 \) lies between \( a \) and \( b \). For definiteness \( f(a) < 0 \) and \( f(b) > 0 \).

a) Choose the first approximation to the root as \( x_1 = \frac{a+b}{2} \). Find \( f(x_1) \). If \( f(x_1) = 0 \) then \( x_1 \) is a root of \( f(x) = 0 \).

b) Suppose \( f(x_1) > 0 \) then the root lies below \( a \). Therefore the second approximation to the root is \( x_2 = \frac{a+x_1}{2} \). Find \( f(x_2) \). If \( f(x_2) = 0 \) then \( x_2 \) is a root of \( f(x) = 0 \).

c) Suppose \( f(x_2) < 0 \) then the root lies above \( x_1 \). Therefore the third approximation to the root is \( x_3 = \frac{x_1+x_2}{2} \).

d) Continue to the above procedure until the root is found to the desired accuracy.

Note: Bisection method is simple but is slow convergence.

3. Find a root of the eq. \( x^3 - 4x - 9 = 0 \), using bisection method correct to 3 decimal places.

\[
\text{Given } x^3 - 4x - 9 = 0
\]

\[ f(x) = x^3 - 4x - 9 \]

Since \( f(x) = -9 < 0 \) and \( f(3) = 6 > 0 \), a root of \( f(x) = 0 \) lies between 2 and 3.

The first approximation to the root is \( x_1 = \frac{2+3}{2} = 2.5 \)

<table>
<thead>
<tr>
<th>#</th>
<th>( x_1 )</th>
<th>( f(x_1) )</th>
<th>Root Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 2.5 )</td>
<td>( f(2.5) = -3.335 &lt; 0 )</td>
<td>((2.5, 3))</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{2.5+3}{2} = 2.75 )</td>
<td>( f(2.75) = 0.799 &gt; 0 )</td>
<td>((2.5, 3))</td>
</tr>
<tr>
<td>3</td>
<td>( 2.75 )</td>
<td>( f(2.75) = -1.42 &lt; 0 )</td>
<td>((2.625, 2.75))</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{2.75+2.625}{2} = 2.6875 )</td>
<td>( f(2.6875) = -0.38 &lt; 0 )</td>
<td>((2.6875, 2.75))</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{2.6875+2.625}{2} = 2.6875 )</td>
<td>( f(2.6875) = 0.225 &lt; 0 )</td>
<td>((2.6875, 2.719))</td>
</tr>
<tr>
<td>6</td>
<td>( \frac{2.6875+2.75}{2} = 2.719 )</td>
<td>( f(2.719) = 0.225 &lt; 0 )</td>
<td>((2.6875, 2.719))</td>
</tr>
<tr>
<td>7</td>
<td>( \frac{2.719+2.75}{2} = 2.737 )</td>
<td>( f(2.737) = 0.13 &lt; 0 )</td>
<td>((2.6875, 2.719))</td>
</tr>
<tr>
<td>8</td>
<td>( \frac{2.737+2.75}{2} = 2.746 )</td>
<td>( f(2.746) = 0.01 &lt; 0 )</td>
<td>((2.6875, 2.719))</td>
</tr>
<tr>
<td>9</td>
<td>( \frac{2.746+2.75}{2} = 2.746 )</td>
<td>( f(2.746) = 0.01 &lt; 0 )</td>
<td>((2.6875, 2.719))</td>
</tr>
</tbody>
</table>

\[ x = 2.746 \]
6. \( x_6 = \frac{2.786 + 2.179}{2} \)
   \(-f(2.704) = -0.045 < 0 \)
   \((2.704, 2.719)\)

4. \( x_7 = \frac{2.704 + 2.712}{2} \)
   \( f(2.712) = 0.099 > 0 \)
   \((2.704, 2.712)\)

8. \( x_8 = \frac{2.704 + 2.708}{2} \)
   \( f(2.708) = 0.026 > 0 \)
   \((2.704, 2.708)\)

9. \( x_9 = \frac{2.704 + 2.708}{2} \)
   \( f(2.706) = -0.009 < 0 \)
   \((2.706, 2.708)\)

10. \( x_{10} = \frac{2.706 + 2.708}{2} \)
    \( f(2.707) = 0.008 > 0 \)
    \((2.706, 2.707)\)

11. \( x_{11} = \frac{2.706 + 2.707}{2} = 2.707 \)

Since \( x_{10} \approx x_{11} \), the req. root is \( x = 2.707 \).

2) Find a two root of \( x e^x = 1 \) which lies between zero \& 1.
   using bisection method.

Given \( e^x \) can be written as \( x e^x - 1 = 0 \)

Let \( -f(x) = x e^x - 1 \)

Since \( f(0) = -1 < 0 \& f(1) = 1.718 > 0 \), a root of \( f(x) = 0 \) lies b/w 0 \& 1.

<table>
<thead>
<tr>
<th>( x_i )</th>
<th>( -f(x_i) )</th>
<th>Root interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ( x_1 = \frac{0+1}{2} = 0.5 )</td>
<td>(-f(0.5) = -0.136 &lt; 0 )</td>
<td>((0.5, 1))</td>
</tr>
<tr>
<td>2 ( x_2 = \frac{0.5+1}{2} = 0.75 )</td>
<td>( f(0.75) = 0.583 &gt; 0 )</td>
<td>((0.5, 0.75))</td>
</tr>
<tr>
<td>3 ( x_3 = \frac{0.5+0.75}{2} = 0.625 )</td>
<td>( f(0.625) = 0.166 &gt; 0 )</td>
<td>((0.5, 0.625))</td>
</tr>
<tr>
<td>4 ( x_4 = \frac{0.5+0.625}{2} = 0.563 )</td>
<td>( f(0.563) = -0.011 &lt; 0 )</td>
<td>((0.563, 0.625))</td>
</tr>
<tr>
<td>5 ( x_5 = \frac{0.563+0.625}{2} = 0.594 )</td>
<td>( f(0.594) = 0.076 &gt; 0 )</td>
<td>((0.563, 0.594))</td>
</tr>
<tr>
<td>6 ( x_6 = \frac{0.563+0.594}{2} = 0.579 )</td>
<td>( f(0.579) = 0.083 &gt; 0 )</td>
<td>((0.563, 0.579))</td>
</tr>
</tbody>
</table>
4. \[ q_7 = \frac{0.568 + 0.571}{2} = 0.569 \quad f(0.569) = 0.011 > 0 \quad (0.563, 0.569) \]
5. \[ q_8 = \frac{0.568 + 0.571}{2} = 0.569 \quad f(0.569) = 0.000 \]
Since \( f(0.569) = 0 \), the req. root is \( q = 0.569 \).

3) Find the real root \( q \) of \( x^3 - 2x - 5 = 0 \) correct to 3 decimals by using bisection method.

Let \( q_1 \) be given \( q_1 = x_1 

Let \( f(x) = x^3 - 2x - 5 \)
Since \( f(x) = -4 < 0 \), \( f(3) = 16 > 0 \), a root of \( f(x) = 0 \) lies \( b/w \) 2 and 3.

1. \[ q_1 = \frac{2 + 3}{2} = 2.5 \quad f(2.5) \approx 5.625 > 0 \quad (2, 2.5) \]
2. \[ q_2 = \frac{2 + 2.5}{2} = 2.25 \quad f(2.25) = 1.891 > 0 \quad (2, 2.25) \]
3. \[ q_3 = \frac{2.25 + 2.25}{2} = 2.25 \quad f(2.25) = 0.546 > 0 \quad (2, 2.25) \]
4. \[ q_4 = \frac{2.25 + 2.25}{2} = 2.25 \quad f(2.25) = 0.546 > 0 \quad (2, 2.25) \]
5. \[ q_5 = \frac{2.25 + 2.25}{2} = 2.25 \quad f(2.25) = 0.546 > 0 \quad (2, 2.25) \]
6. \[ q_6 = \frac{2.094 + 2.125}{2} = 0.1094 < 0 \quad f(2.094) = 0.0066 > 0 \quad (2.094, 2.110) \]
7. \[ q_7 = \frac{2.094 + 2.125}{2} = 2.109 \quad f(2.109) = 0.0174 > 0 \quad (2.094, 2.110) \]
8. \[ q_8 = \frac{2.094 + 2.109}{2} = 2.098 \quad f(2.098) = 0.039 > 0 \quad (2.094, 2.104) \]
9. \[ q_9 = \frac{2.094 + 2.098}{2} = 2.096 \quad f(2.096) = 0.016 > 0 \quad (2.094, 2.096) \]
10. \[ q_{10} = \frac{2.094 + 2.096}{2} = 2.095 \quad f(2.095) = 0.005 > 0 \quad (2.094, 2.096) \]
11. \[ q_{11} = \frac{2.094 + 2.095}{2} = 2.095 \quad f(2.095) = 0.005 > 0 \quad (2.094, 2.095) \]

Therefore, the req. root is 2.095.
Regula-falsi method (Method of false position):

This is the oldest method to find the real root of \( f(x) = 0 \) and closely resembles the bisection method.

To find the approximate root of the equation \( f(x) = 0 \) in the interval \((a, b)\), assume that \( f(a) < 0, f(b) > 0 \).

Geometrically, this method is equivalent to replacing the curve \( y = f(x) \) by a chord that passes through the point \( A(a, f(a)) \) and \( B(b, f(b)) \). The equation of the chord \( AB \) is \( \frac{y - f(a)}{f(b) - f(a)} = \frac{x - a}{b - a} \).

Suppose the chord \( AB \) meets the \( x \)-axis at \( x = x_1 \); put \( x = x_1 \),

\[
y = 0 \text{ in eq } 1 \quad \Rightarrow \quad f(a) = \frac{f(b) - f(a)}{b - a} (x_1 - a)
\]

\[
\Rightarrow \quad (x_1 - a) = \frac{-(b-a)f(a)}{f(b) - f(a)} \Rightarrow x_1 = a + \frac{-(b-a)f(a)}{f(b) - f(a)}
\]

\[
\Rightarrow x_1 = a + \frac{af(b) - a f(a) + b f(a) - af(a)}{f(b) - f(a)} \Rightarrow x_1 = \frac{af(b) - bf(a) + af(a)}{f(b) - f(a)} \tag{2}
\]

which is the first approximation to the root.

Find \( f(x_1) \). If \( f(x_1) = 0 \) then \( x_1 \) is root of \( f(x) = 0 \).

Suppose \( f(x_1) > 0 \) then a root of \( f(x) = 0 \) lies below \( a \) and \( x_1 \).

Replace '\( b \)' by \( x_1 \) in eq (2), we get the second approximation to the root as \( x_2 \), i.e.,

\[
x_2 = a + \frac{af(x_1) - af(a)}{f(x_1) - f(a)} \tag{3}
\]

Find \( f(x_2) \). If \( f(x_2) = 0 \) then \( x_2 \) is root of \( f(x) = 0 \).

Suppose \( f(x_2) < 0 \) then a root of \( f(x) = 0 \) lies below \( x_1 \) and \( x_2 \).

Replace '\( a \)' by \( x_2 \) in eq (3), we get the third approximation to the root as \( x_3 \), i.e,

\[
x_3 = a + \frac{af(x_2) - af(x_1)}{f(x_2) - f(x_1)}
\]

Continue the above procedure until the root is found to the desired accuracy.
1. Find a real root \( q \) of the eq. \( ax^4 + x - 1 = 0 \) by the method of false position correct to four decimal places.

Given \( a^3 + x - 1 = 0 \)

Let \( f(x) = ax^4 + x - 1 \)

Since \( f(0) = -1 < 0 \) and \( f(1) = 1 > 0 \), a root of \( f(x) = 0 \) lies b/w 0 & 1.

By the method of false position, the first approximation to the root.

\[ x_1 = \frac{0f(1) - f(0)}{f(1) - f(0)} = \frac{0 - 1(-1)}{1 - (-1)} = 0.5 \]

\( f(0.5) = -0.3125 < 0 \). So root lies b/w 0.5 & 1.

\[ x_2 = \frac{0.5f(1) - f(0.5)}{f(1) - f(0.5)} = 0.6364 \]

\( f(0.6364) = -0.0106 < 0 \). So root lies b/w 0.6364 & 1.

\[ x_8 = 0.6364 \]

\( f(0.6364) = -0.0001 < 0 \). So root lies b/w 0.6364 & 1.

\[ x_9 = 0.6364 \]

\( f(0.6364) = -0.0001 < 0 \). So root lies b/w 0.6364 & 1.

\[ x_8 = 0.6364 \]

Since \( x_9 = x_8 \), the req. root is \( x = 0.6364 \) correct to 4 decimal places.

2. Determine the root of \( x e^x - 2 = 0 \) by regula-falsi.

Given eq. is \( x e^x - 2 = 0 \)

Let \( f(x) = x e^x - 2 \)
5. Use method of false position to find the 4th root of 32 correct to 8 decimal places.

So let \( \alpha = \sqrt[4]{32} \) so that \( \alpha^4 = 32 \Rightarrow \alpha^4 - 32 = 0 \)

Take \( f(x) = x^4 - 32 \)

\( f(2) = 2 - 0 < 0 \) and \( f(3) = 81 > 0 \), a root \( \Rightarrow f(x) = 0 \).

By the method of false position we have \( \alpha_1 = \frac{2 \cdot f(3) - 3 \cdot f(2)}{f(3) - f(2)} = 2.246 \)

\( f(2.246) = 0.296 < 0 \), so that root lies below 2.246 \& 3

\( \alpha_2 = \frac{3 \cdot f(2.246) - 2 \cdot f(2)}{f(3) - f(2.246)} = 2.335 \)

\( f(2.335) = -0.234 < 0 \), so that root lies below 2.335 \& 3

\( \alpha_3 = \frac{3 \cdot f(2.335) - 2 \cdot f(2.335)}{f(3) - f(2.335)} = 2.366 \)

\( f(2.366) = 0.016 < 0 \), so that root lies below 2.366 \& 3

\( \alpha_4 = \frac{3 \cdot f(2.366) - 2 \cdot f(2.366)}{f(3) - f(2.366)} = 2.374 \)

\( f(2.374) = -0.076 < 0 \), so that root lies below 2.374 \& 3

\( \alpha_5 = \frac{3 \cdot f(2.374) - 2 \cdot f(2.374)}{f(3) - f(2.374)} = 2.378 \)

Since \( \alpha_5 \) is the req root is 2.378.

Hence the fourth root of 32 to 3 decimals is 2.378.

6. Using regula falsi method find an approximate root of \( x^5 - 4x + 1 = 0 \)

Given eq is \( x^5 - 4x + 1 = 0 \)

Let \( f(x) = x^5 - 4x + 1 \)

Since \( f(0) = 1 > 0 \), \( f(1) = -1 < 0 \), a root of \( f(x) = 0 \) lies below 0 \& 1

By regula falsi method the first approximation to the root is \( \alpha_1 = \frac{af(1) - f(0)}{f(1) - f(0)} = 0.333 \)

\( f(0.333) = -0.2962 < 0 \), so the root lies below 0 \& 0.333.
\[ x_1 = \frac{1}{f(x_0)} = 0.2541 \]

\[ f(0.2541) = -0.014 < 0, \text{ so the root lies below } 0 \text{ at } x_1 = 0.2541 \]

\[ x_2 = \frac{1}{f(x_1)} = 0.2542 \]

\[ f(0.2542) = -0.0007, \text{ so the root lies below } 0 \text{ at } x_2 = 0.2542 \]

\[ x_3 = \frac{1}{f(x_2)} = 0.2541 \]

\[ f(0.2541) = 0.0000 \]

The root is 0.2541 correct to 4 decimals.

*Theorem:

Let \( x = \alpha \) be a root of \( f(x) = 0 \) and \((a, b)\) be any interval containing \( \alpha \). Suppose \( f(x) < 0 \) on \((a, b)\).

If \( f(x) < 0 \) on \([a, b] \) then the sequence of approximations \( x_0, x_1, \ldots, x_n \) will converge to the root \( \alpha \), provided the initial approximation \( x_0 \) is chosen from \((a, b)\).

*Iteration method (Fixed point iteration method)*

Choose two points \( a \) and \( b \) such that \( f(a)f(b) < 0 \) then the root \( \alpha \), \( f(x) = 0 \) lies between \( a \) and \( b \).

Choose \( x_0 = \frac{a + b}{2} \).

Rewrite the given eq. \( f(x) = 0 \) in the form \( x = \phi(x) \) to get

where \( |\phi'(x)| < 1 \) on \([a, b] \).

Put \( x = x_0 \) in the R.H.S of eq \( (1) \) we get the 1st approximation to the root as \( x_1 = \phi(x_0) \).

Again put \( x = x_1 \) in R.H.S of eq \( (1) \) we get the 2nd approximation to the root as \( x_2 = \phi(x_1) \).

Proceeding in the same way, the successive approximations
are given by \( a_3 = \phi(x_1), a_4 = \phi(x_3), \ldots, a_n = \phi(x_{n-1}) \) — (2)

which is known as iteration formula. \[ n = 1, 2, \ldots, \]

4. Find the real root of \( x^2 - x - 1 = 0 \) correct to four decimals places by iteration method.

Given eq. is \( x^2 - x - 1 = 0 \) — (1)

Take \( f(x) = x^2 - x - 1 \)

Since \( f(1) = -1 < 0 \) and \( f(2) = 5 > 0 \), a root of \( f(x) = 0 \) lies 1 \& 2

Rewriting eq (1) as \( a^2 = a + 1 \) \( \Rightarrow a = (a + 1)^{1/2} \)

It is of the form \( a = \phi(x) \) where \( \phi(x) = (x + 1)^{1/2} \)

\[ \phi'(x) = \frac{1}{2}(x + 1)^{-1/2} \]

Clearly, \( |\phi'(x)| = \frac{1}{2} \left| \frac{1}{(x + 1)^{1/2}} \right| < 1 \text{ for } 1 \leq x \leq 2 \)

Iteration method is applicable

choose initial approximation \( x_0 = \frac{1 + \sqrt{5}}{2} = 1.6 \)

By iteration method we have \( x_n = \phi(x_{n-1}) \)

\[ x_n = (x_{n-1} + 1)^{1/2} \] — (2)

Put \( n = 1, 2, 3, 4, \ldots \) in (2) we get

\[ x_1 = (x_0 + 1)^{1/2} = (1.6 + 1)^{1/2} = 1.3828 \]

\[ x_2 = (x_1 + 1)^{1/2} = (1.3828 + 1)^{1/2} = 1.839 \]

\[ x_3 = (x_2 + 1)^{1/2} = (1.839 + 1)^{1/2} = 1.8269 \]

\[ x_4 = (x_3 + 1)^{1/2} = (1.8269 + 1)^{1/2} = 1.8249 \]

\[ x_5 = (x_4 + 1)^{1/2} = (1.8249 + 1)^{1/2} = 1.8248 \]

\[ x_6 = (x_5 + 1)^{1/2} = (1.8248 + 1)^{1/2} = 1.8247 \]

\[ x_7 = (x_6 + 1)^{1/2} = (1.8247 + 1)^{1/2} = 1.8247 \]

Since \( x_6 \approx x_7 \), the real root is \( 1.8247 \)
Find the +ve root of 
\[ x^3 = 8x + 5 \]
which lies near \( x = 2 \).

Given eq. \( x^3 = 8x + 5 \) or \( x^3 - 8x - 5 = 0 \)

Take \( f(x) = x^3 - 8x - 5 \)

Since \( f(2) = -1 < 0 \) and \( f(3) = 16 > 0 \), a root of \( f(x) = 0 \) lies b/n 2 & 3.

Rewriting eqn (1) as \( x = (2x + 5)^{1/3} \)

It is of form \( x = \phi(x) \) where \( \phi(x) = (2x + 5)^{1/3} \)

\[ \Rightarrow \phi(x) = \frac{1}{3} \left( \frac{1}{(2x + 5)^{2/3}} \right) < 1 \text{ for } x \leq 2 \]

:. Iteration method is applicable choose \( x_0 = \frac{9 + 5}{2} = 2.5 \)

By iteration method, we have \( x_n = \phi(x_{n-1}) \)

\[ x_0 = (2x_0 + 5)^{1/3} = (2 \times 2.5 + 5)^{1/3} = 2.1544 \]

\[ x_1 = (2x_1 + 5)^{1/3} = (2 \times 2.1544 + 5)^{1/3} = 2.1086 \]

\[ x_2 = (2x_2 + 5)^{1/3} = (2 \times 2.1086 + 5)^{1/3} = 2.0959 \]

\[ x_3 = (2x_3 + 5)^{1/3} = (2 \times 2.0959 + 5)^{1/3} = 2.0948 \]

\[ x_4 = (2x_4 + 5)^{1/3} = (2 \times 2.0948 + 5)^{1/3} = 2.0946 \]

\[ x_5 = (2x_5 + 5)^{1/3} = (2 \times 2.0946 + 5)^{1/3} = 2.0946 \]

\[ x_5 = x_6 \text{, the req root is } 2.0946 \]

6. Use method of iteration to find a +ve root of 
\[ \sqrt{e^x} - 1 = 0 \]

Correct to 3 decimals.

Given eqn can be written as \( x = e^x \)

It is of the form \( x = \phi(x) \) where \( \phi(x) = e^x \) \[ \phi(x) = -e^{-x} \]
clearly, \( |g'(x)| = \frac{1}{1 + x^2} < 1 \) at \( 0 \leq x \leq 1 \)

\[ \text{Iteration method is possible or applicable} \]

\[ \text{choose}, \quad x_0 = \frac{0 + 1}{2} = 0.5 \]

By the method of iteration method, we have \( x_n = g(x_{n-1}) \)

i.e., \( x_n = (e^{-x_{n-1}}) \) — (2)

Put \( n = 1, 2, 3, \ldots \) in (2), we get

\[ x_1 = e^{-x_0} = e^{-0.5} = 0.6065 \]
\[ x_2 = e^{-x_1} = e^{-0.6065} = 0.545 \]
\[ x_3 = e^{-x_2} = e^{-0.545} = 0.580 \]
\[ x_4 = e^{-x_3} = e^{-0.580} = 0.560 \]
\[ x_5 = e^{-x_4} = e^{-0.560} = 0.571 \]
\[ x_6 = e^{-x_5} = e^{-0.571} = 0.555 \]
\[ x_7 = 0.568 \]

Since \( x_7 \approx x_8 \), the req. root, is \( 0.568 \)

* Newton-Raphson Method (Newton’s Iterative Method):

Let \( x_0 \) be an approximation to the root of \( f(x) = 0 \). If \( x_1 = x_0 + h - c \)

where \( h \) is very small, be the root of \( f(x) = 0 \) then \( f(x_1) = 0 \) i.e.

\[ f(x_0 + h) = 0 \] — (2) expanding \( f(x_0 + h) \) by Taylor’s series, eq. (2) becomes

\[ f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots + \frac{h^n}{n!} f^{(n)}(x_0) + \cdots = 0 \] — (3) since \( h \) is small,

Neglecting the higher powers of \( h \) i.e., \( h^2, h^3, \ldots \), eq. (3) becomes

\[ f(x_0) + h f'(x_0) = 0 \text{ or } h = \frac{-f(x_0)}{f'(x_0)} \]

Substituting \( h \) value in eq. (1) we get

\[ x_1 = x_0 + \frac{-f(x_0)}{f'(x_0)} \] — (4) which is the first approximation to the root.

Again, starting with \( x_1 \) & assuming \( x_2 = x_1 + h \) to be the root of

we get the second approximation to the root as

\[ x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \] — (5)

The third approximation to the root is

\[ x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \] — (6)

In general, the \( n \)th approximation to the root is

which is known as ‘Newton-Raphson method’ or ‘Newton’s Iteration method’.

Note: choose initial approximation \( x_0 \approx a \), where the root &

\( f(x_0) \approx 0 \) lies btw \( a \) & \( b \).
1. Find the root of \( ax^4 - x - 10 = 0 \) correct to 3 decimal places.

Given eq. is \( ax^4 - x - 10 = 0 \).

Let \( f(x) = ax^4 - x - 10 \) so that \( f'(x) = 4ax^3 \).

Since \( f(1) = -10 < 0 \) and \( f(2) = 4 > 0 \), a root of \( f(x) = 0 \) lies b/w 1 & 2.

Choose initial approximation \( x_0 = \frac{1 + 2}{2} = 1.5 \).

By Newton-Raphson iteration formula, we have \( x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \).

\[
x_n = x_{n-1} - \frac{x_{n-1}^5 - x_{n-1} - 10}{4x_{n-1}^4 - 1} = \frac{4x_{n-1}^4 - x_{n-1}^5 - x_{n-1}^4 + x_{n-1} + 10}{4x_{n-1}^4 - 1}.
\]

\[\text{Eq. } x_n = \frac{8x_{n-1}^4 + 10}{4x_{n-1}^4 + 1} \quad \text{(1)} \]

Thus, \( x_1 = \frac{3x_0^4 + 10}{4x_0^4 - 1} = \frac{3(1.5)^4 + 10}{4(1.5)^4 - 1} = 2.015 \)

\( x_2 = \frac{3x_1^4 + 10}{4x_1^4 - 1} = \frac{3(2.015)^4 + 10}{4(2.015)^4 - 1} = 1.874 \)

\( x_3 = \frac{3x_2^4 + 10}{4x_2^4 - 1} = \frac{3(1.874)^4 + 10}{4(1.874)^4 - 1} = 1.856 \)

\( x_4 = \frac{3x_3^4 + 10}{4x_3^4 - 1} = \frac{3(1.856)^4 + 10}{4(1.856)^4 - 1} = 1.856 \)

Since \( x_3 \approx x_4 \), the req. root is 1.856.

2. Find the real root of \( x^5 + 2x^2 + 0.4 = 0 \) using Newton-Raphson.

Given eq. is \( x^5 + 2x^2 + 0.4 = 0 \).

Take \( f(x) = x^5 + 2x^2 + 0.4 \) so that \( f'(x) = 5x^4 + 4x \).

Since \( f(-1.5) = -8.6 < 0 \) and \( f(-2) = 0.4 > 0 \), a root of \( f(x) = 0 \) lies b/w -3 & -2.

Choose initial approximation \( x_0 = \frac{-3 - 2}{2} = -2.5 \).

By Newton-Raphson iteration formula, we have \( x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \).

\[
x_n = x_{n-1} - \frac{x_{n-1}^5 + 2x_{n-1}^2 + 0.4}{5x_{n-1}^4 + 4x_{n-1}} = \frac{5x_{n-1}^5 + 4x_{n-1}^2 - x_{n-1}^4 - 2x_{n-1} - 0.4}{5x_{n-1}^4 + 4x_{n-1}}.
\]

\[\text{Eq. } x_n = \frac{8x_{n-1}^5 + 2x_{n-1}^2 - 0.4}{5x_{n-1}^4 + 4x_{n-1}} = \frac{2(3.5)^5 + 2(-3.5)^2 - 0.4}{5(-3.5)^4 + 4(-3.5)} = -2.016386 \]
\( a_3 = \frac{8x_1^3 + 2x_1^2 - 0.4}{3x_1^2 + 4x_1} = \frac{2(-2.189)^3 + 2(-2.189)^2 - 0.4}{8(-2.189)^2 + 4(-2.189)} = -2.0990 \)

\( a_3 = \frac{8x_2^3 + 2x_2^2 - 0.4}{3x_2^2 + 4x_2} = \frac{2(-2.099)^3 + 2(-2.099)^2 - 0.4}{3(-2.099)^2 + 4(-2.099)} = -2.0915 \)

\( a_4 = \frac{8x_3^3 + 2x_3^2 - 0.4}{3x_3^2 + 4x_3} = \frac{2(-2.0915)^3 + 2(-2.0915)^2 - 0.4}{3(-2.0915)^2 + 4(-2.0915)} = -2.0914 \)

\( a_5 = \frac{8x_4^3 + 2x_4^2 - 0.4}{3x_4^2 + 4x_4} = \frac{2(-2.0914)^3 + 2(-2.0914)^2 - 0.4}{3(-2.0914)^2 + 4(-2.0914)} = -2.0914 \)

Since \( a_4 = a_5 \), the req. root is \(-2.0914\).

3. Using Newton-Raphson method, find the real root of \( x \log_{10} x = 1.2 \) correct to 5 decimals.

Given \( x \log_{10} x = 1.2 \) \( \Rightarrow \) \( x \log_{10} x - 1.2 = 0 \)

Let \( f(x) = x \log_{10} x - 1.2 \), so that \( f'(x) = 1 \times \log_{10} x + x \times \frac{1}{x} \times \frac{1}{x} \times \log_e = \frac{1}{10} \log_e \)

Since \( f(2) = 0.59744 \lt 0 \) and \( f(3) = 0.23136 \gt 0 \), \( \frac{d}{dx} (\log_{10} x) = \frac{1}{x} \log_e \)

A root of \( f(x) = 0 \) lies between 2 & 3.

Above \( x_0 = \frac{2 + 3}{2} = 2.5 \)

By Newton-Raphson method, we have \( x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \)

\( x_n = x_{n-1} - \frac{\log_{10} x_{n-1} - 1.2}{\log_{10} x_{n-1} + 0.43429} = \frac{x_{n-1} \log_{10} x_{n-1} + 0.43429 x_{n-1} - x_{n-1} \log_{10} x_{n-1} + 1.2}{\log_{10} x_{n-1} + 0.43429} \)

\( \therefore x_n = 0.43429 x_{n-1} + 1.2 \) \( \log_{10} x_{n-1} + 0.43429 \)

Put \( n = 1, 2, \ldots, 9 \) \( (1) \) we get

\( x_1 = \frac{0.43429 x_0 + 1.2}{\log_{10} x_0 + 0.43429} = \frac{0.43429(2.5) + 1.2}{\log_{10}(2.5) + 0.43429} = 2.74651 \)

\( x_2 = \frac{0.43429 x_1 + 1.2}{\log_{10} x_1 + 0.43429} = 2.74065 \)

\( x_3 = \frac{0.43429 x_2 + 1.2}{\log_{10} x_2 + 0.43429} = 2.74065 \)

Since \( x_3 = x_2 \), the req. root is \( 2.74065 \).
3.2. INTERPOLATION

Finite differences:

Suppose the experimental data for an unknown function \( y=f(x) \) for a set of \( n+1 \) values is given below.

\[
x = x_0 \ x_1 \ x_2 \ \ldots \ x_{n-1} \ x_n \\
y = f(x) \quad y_0 \ y_1 \ y_2 \ \ldots \ y_{n-1} \ y_n
\]

The process of determining the value of \( y \) for a value of \( x \), between \( x_0 \) & \( x_n \) is known as interpolation, where \( x_0, x_1, \ldots, x_n \) are called interpolation points. In extrapolation the value of \( y \) is determined for a value of \( x \) but generally interpolation includes extrapolation also. The study of the interpolation is based on the concept of differences of a function which we proceed to discuss.

Suppose the function \( y=f(x) \) is calculated for a equally spaced values \( x=x_0, x_0+h, x_0+2h, \ldots, x_0+nh \) giving \( y=y_0, y_1, \ldots, y_n \), where \( h \) is the interval of differencing. To determine the values of \( f(x) \) for some intermediate value \( x \), the following 3 types of differences are found useful.

- **Forward differences**: The differences \( y_1-y_0, y_2-y_1, \ldots, y_n-y_{n-1} \) denoted by \( \Delta y_0, \Delta y_1, \ldots, \Delta y_{n-1} \) respectively are called first forward differences. Here \( \Delta \) is the forward difference operator. Thus

\[
\Delta y_Y = \Delta y_{Y+1} - \Delta y_Y \quad Y=0, 1, 2, \ldots, n-1
\]

Similarly, the second forward differences are defined by

\[
\Delta^2 y_Y = \Delta y_{Y+1} - \Delta y_Y
\]

In general, \( \Delta^m y_Y = \Delta^{m-1} y_{Y+1} - \Delta^{m-1} y_Y \) defines the \( m^{th} \) forward difference.

- **Backward differences**: The differences \( y_1-y_0, y_2-y_1, \ldots, y_n-y_{n-1} \) denoted by \( \nabla y_1, \nabla y_2, \ldots, \nabla y_n \) respectively are called first backward differences.
Here $\nabla$ is the backward differences operator.

Thus $\nabla y_i = y_i - y_{i-1}$; $i = 1, 2, 3, \ldots \n$

i) The $n$th backward differences are given by $\nabla^n y_i = \nabla^{n-1} y_i - \nabla^{n-1} y_{i-1}$;

$\ldots n = 0, 1, 2, \ldots \n$

iii) Central differences: The differences $y_{i+1} - y_i \ldots y_0 - y_{-1}$ denoted by $\delta y_1, \delta y_2, \ldots, \delta y_{n-1}$ respectively, are called first central differences. Here $\delta$ is the central difference operator.

Some useful operators are:

Suppose that the function $y = f(x)$ is defined for equally spaced values $x_i = x_0 + ih; i = 1, 2, 3, \ldots n$, where $h$ is interval of difference.

i) $\Delta$: The forward difference operator denoted by $\Delta$ (read as delta) is defined as $\Delta f(x) = f(x+h) - f(x)$

ii) $\nabla$: The backward difference operator denoted by $\nabla$ (read as nabla) is defined as $\nabla f(x) = f(x) - f(x-h)$

iii) $\epsilon$: The shift operator denoted by $\epsilon$ is defined as $\epsilon f(x) = f(x+h)$

iv) Note: $e^0 f(x) = f(x+0h)$, or $e^0 = 1$

$e^h f(x) = f(x+4h)$, $e^{-h} f(x) = f(x-h)$

iv) $\delta$: The central difference operator is denoted by $\delta$ is defined as $\delta f(x) = f(x+h) - f(x-h)$

v) $\mu$: The mean or averaging operator denoted by $\mu$ is defined as $\mu f(x) = \frac{1}{2} \left[ f(x+h) + f(x-h) \right]$

Relation between the operators $\Delta, \nabla, \epsilon, \delta, \mu$:

We shall now establish the following identities:

i) $\Delta = \epsilon - 1$  ii) $\nabla = 1 - \epsilon^{-1}$  iii) $\delta = e^{h/2} - e^{-h/2}$  iv) $\mu = \frac{1}{2} \left[ e^{h/2} + e^{-h/2} \right]$

v) $\epsilon = e^{hD}$ where $D = \frac{d}{dx}$
\textbf{Proof:}  
1) we have
\[ \Delta f(x) = f(x+h) - f(x) = e^{f(x)} - e^{-f(x)} \]
\[ \Delta f(x) = (e-1) f(x) \]
\[ \therefore \Delta = e-1 \]

ii) we have
\[ \tau f(x) = f(x) - f(x-n) \]
\[ = f(x) - e^{-n} f(x) = (1-e^{-n}) f(x) \]
\[ \therefore \tau = (1-e^{-n}) \]

iii) we have
\[ s f(x) = f(x+h/2) - f(x-h/2) \]
\[ = e^{h/2} f(x) - e^{-h/2} f(x) = (e^{h/2} - e^{-h/2}) f(x) \]
\[ \therefore s = e^{h/2} - e^{-h/2} \]

iv) we have
\[ \mu f(x) = \frac{1}{2} \left[ f(x+h/2) + f(x-h/2) \right] \]
\[ = \frac{1}{2} \left[ e^{h/2} f(x) + e^{-h/2} f(x) \right] = \frac{1}{2} \left[ e^{h/2} + e^{-h/2} \right] f(x) \]
\[ \therefore \mu = \frac{1}{2} [e^{h/2} + e^{-h/2}] \]

v) we have
\[ \epsilon f(x) = f(x+h) \]
\[ = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots \ldots \ldots \]
\[ = f(x) + \frac{h}{1!} D f(x) + \frac{h^2}{2!} D^2 f(x) + \frac{h^3}{3!} D^3 f(x) + \ldots \ldots \] (by Taylor's series expansion)
\[ = \left[ 1 + \frac{hD}{1!} + \frac{h^2 D^2}{2!} + \frac{h^3 D^3}{3!} + \ldots \right] f(x) \]
\[ = \left[ 1 + \frac{hD}{1!} + \frac{h^2 D^2}{2!} + \frac{h^3 D^3}{3!} + \ldots \right] f(x) = e^{hD} f(x), \text{ where } D = \frac{d}{dx} \]
\[ \therefore \epsilon = e^{hD} \]

Prove the following:
1. \[ e^{h/2} = \mu + \frac{\epsilon}{2} \]

R.H.S = \[ \mu + \frac{\epsilon}{2} \]
\[ = \frac{1}{2} \left( e^{h/2} + e^{-h/2} \right) + \frac{1}{2} \left( e^{h/2} - e^{-h/2} \right) \]
\[ = \frac{1}{2} \left( 2 e^{h/2} \right) = e^{h/2} = \text{L.H.S} \]
\[ \therefore e^{h/2} = \mu + \frac{\epsilon}{2} \]

2. \[ \tau = \delta e^{-h/2} \]

R.H.S = \[ \delta e^{-h/2} \]
\[ = e^{h/2} e^{-h/2} - e^{-h/2} e^{-h/2} \]
\[ = e^0 - e^{-1} \]
\[ = 1 - e^{-1} \]
\[ = \tau = \text{L.H.S} \]
\[ \therefore \tau = 1 - e^{-1} \]
We now state that two important interpolation formula by means of the forward and backward differences of a function.

Let the function \( y = f(x) \) take the values \( y_0, y_1, \ldots, y_n \) corresponding to the values \( x_0, x_0 + h, x_0 + 2h, \ldots, x_0 + nh \) of \( x \).

**Newton's Forward Interpolation Formula (NFIF):**

Suppose it is required to evaluate the value of \( y \) for \( x = x_0 + ph \) where \( p \) is any real no. Then \( y(x) = y_0 + \frac{p}{1!} \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \ldots + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \ldots + \frac{p(p-1)(p-2)\ldots(p-n+1)}{n!} \Delta^n y_0 \) where \( \rho = \frac{x-x_0}{h} \).

**Newton's Backward Interpolation Formula (NBIF):**

Suppose it is required to evaluate the value of \( y \) for \( x = x_0 + ph \) where \( p \) is any real no. Then \( y(x) = y_n + \frac{p}{1!} \Delta y_n + \frac{p(p+1)}{2!} \Delta^2 y_n + \ldots + \frac{p(p+1)(p+2)\ldots(p+n-1)}{n!} \Delta^n y_n \) where \( \rho = \frac{x-x_0}{h} \).

**Note:**

1. NFIF is used to interpolate the values of \( y \) near beginning of the table and extrapolate the values of \( y \) just left to the beginning of the table.
2. NBIF is used to interpolate the values of \( y \) near the end of the table and extrapolate the values of \( y \) just right to the end of the table.

**Example:**

The population of a town in decennial census was given below:

<table>
<thead>
<tr>
<th>Year</th>
<th>1921</th>
<th>1931</th>
<th>1941</th>
<th>1951</th>
<th>1961</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population (in thousand)</td>
<td>46</td>
<td>66</td>
<td>81</td>
<td>93</td>
<td>101</td>
</tr>
</tbody>
</table>
6. The area of circle with diameter 105 cm is given. For the following values:

\[ x = 0, 50, 65, 70, 75, 80, 85, 90, 95, 100 \]

Calculate the area of a circle whose diameter is 105 cm.

 Ans: 8666 sq cm

The difference table is given below:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>( \Delta y )</th>
<th>( \Delta^2 y )</th>
<th>( \Delta^3 y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>648</td>
<td>40</td>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>85</td>
<td>684</td>
<td>38</td>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>90</td>
<td>726</td>
<td>40</td>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>95</td>
<td>766</td>
<td>40</td>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>100</td>
<td>804</td>
<td>40</td>
<td>-2</td>
<td>4</td>
</tr>
</tbody>
</table>

Here \( x = 105 \), \( x_0 = 100 \), \( h = 5 \) such that \( p = \frac{x-x_0}{h} = \frac{105-100}{5} = 1 \)

Using NBF, we get:

\[ y(x) = y_0 + \frac{p}{1!} \Delta y_0 + \frac{p(p+1)}{2!} \Delta^2 y_0 + \frac{p(p+1)(p+2)}{3!} \Delta^3 y_0 \]

\[ y(105) = 7854 + 1(166) + \frac{1}{2} (40) + \frac{1}{6} \cdot 4 = 7854 + 1766 + 20 + 6 \]

\[ y(105) = 8666 \]

To fit a polynomial of degree 8 which takes the following values:

\[ x = 3 \quad 4 \quad 5 \quad 6 \]

\[ y = 6 \quad 24 \quad 60 \quad 120 \]

The difference table is given below:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>( \Delta y )</th>
<th>( \Delta^2 y )</th>
<th>( \Delta^3 y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>6</td>
<td>18</td>
<td>18</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>24</td>
<td>36</td>
<td>24</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>60</td>
<td>36</td>
<td>24</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>120</td>
<td>60</td>
<td>24</td>
<td></td>
</tr>
</tbody>
</table>

Here \( x = 3 \), \( x_0 = 3 \), \( h = 1 \) so that \( p = \frac{x-x_0}{h} = \frac{3-3}{1} = 0 \)

Using NBF we get:

\[ y(x) = y_0 + \frac{p}{1!} \Delta y_0 + \frac{p(p+1)}{2!} \Delta^2 y_0 \]

\[ y = 6 + (x-3)(18) + (x-3)(x-4)(18) + \frac{(x-3)(x-4)(x-5)}{6} \]

\[ y = 6 + 18x - 54 + (x-3)(x-4)(18) + (x-3)(x-4)(x-5) \]

\[ y = x^8 - 3x^2 + 2x \]
8. Find the number of men getting wages below ₹15 from the following data: Wages (in ₹): 0-10, 10-20, 20-30, 30-40, 40-50.
Frequency: 9, 30, 35, 42, 28
\[ \text{Mean} = \frac{28+5+35+42+9}{2} = 34 \]

Sei: The difference table is given below:

<table>
<thead>
<tr>
<th>x</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>9</td>
<td>89</td>
<td>116</td>
<td></td>
</tr>
</tbody>
</table>

\[ \Delta y \quad \Delta^2 y \quad \Delta^3 y \]

10 | 9   | 30          | (\Delta^2 y) |
20 | 89  | 30          |
30 | 116 |             |
40 |     |             |

\[ \Delta = 30 - 116 = -86 \]

\[ \text{Here } x_0 = 10, \Delta x = 15, h = 10 \text{ so that } P = \frac{x-x_0}{h} = \frac{15-10}{10} = 0.5 \]

Using NFIF we get:

\[ y(x) = y_0 + \frac{P}{11!} \Delta y_0 + \frac{P(P-1)}{8!} \Delta^2 y_0 + \frac{P(P-1)(P-2)}{6!} \Delta^3 y_0 \]

\[ y(15) = 9 + (0.5)(20) + (0.5)(-0.5)(9) + (0.5)(-0.5)(-1.5)(0) \]

\[ y(15) = 23.5 \]

\[ y(15) \approx 24 \]

*Interpolation with unequal intervals:

Newton's interpolation formula is applicable only to data with equal intervals. In this section, we discuss Lagrange interpolation formula which is best suited for data with unequal intervals.

If \( y = f(x) \) takes the values \( y_0, y_1, y_2, \ldots, y_n \) corresponding to \( x_0, x_1, x_2, \ldots, x_{n-1}, x_n \) (not necessarily equidistant) then:

\[ f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\ldots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1})} y_n \]

This is known as Lagrange's interpolation formula.
Find the value of \( \log_{10} 656 \).

Using Lagrange's Interpolation formula, we get:

\[
y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3
\]

Put \( x = 656 \), we get:

\[
y(656) = \frac{(-2)(-3)(-5)}{(-4)(-5)(-6)} \times 2.8182 + \frac{(-2)(-3)(-5)}{(-4)(-5)(-6)} \times 2.8182 + \frac{(-2)(-3)(-5)}{(-4)(-5)(-6)} \times 2.8182 + \frac{(-2)(-3)(-5)}{(-4)(-5)(-6)} \times 2.8182
\]

\[
y(656) = 2.8182
\]

ie \( \log_{10} 656 = 2.8182 \)

Find the polynomial \( f(x) \) by using Lagrange's formula and hence find \( f(3) \) for \( x_i = 0, 1, 2, 3 \).

\[
f(x) = \sum_{i=0}^{3} \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_i-x_0)(x_i-x_1)(x_i-x_2)} y_i
\]

\[
f(3) = \frac{(3-0)(3-1)(3-2)}{(0-0)(0-1)(0-2)} y_0 + \frac{(3-0)(3-1)(3-2)}{(1-0)(1-1)(1-2)} y_1 + \frac{(3-0)(3-1)(3-2)}{(2-0)(2-1)(2-2)} y_2 + \frac{(3-0)(3-1)(3-2)}{(3-0)(3-1)(3-2)} y_3
\]

\[
f(3) = \frac{(2)\times(1)\times(1)}{(0)\times(-1)\times(-2)} \times 2 + \frac{(2)\times(1)\times(1)}{(1)\times(1)\times(0)} \times 3 + \frac{(2)\times(1)\times(1)}{(2)\times(2)\times(1)} \times 5 + \frac{(2)\times(1)\times(1)}{(3)\times(0)\times(-1)} \times 6
\]

\[
f(3) = 2 \times (2^2 + 2) = 12
\]

\[
f(3) = 12
\]

\[
f(x) = \frac{x^3 + 12 - x}{20}
\]

\[
f(x) = \frac{x^3 + 12 - x}{20}
\]

\[
f(3) = \frac{3^3 + 12 - 3}{20}
\]

\[
f(3) = \frac{35}{20} = 1.75
\]
UNIT-IV

NUMERICAL DIFFERENTIATION, INTEGRATION & CURVE FITTING
NUMERICAL DIFFERENTIATION AND INTEGRATION

- Numerical differentiation:

It is the process of evaluating a derivative of an unknown function \( y = f(x) \) at some intermediary value of \( x \), from the given set of values \((x_i, y_i)\); \( i = 0, 1, 2, \ldots, n \) where the values of \( x_i \) are equispaced.

Suppose it is required to evaluate \( \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \) at some point near the beginning of the table; use Newton's forward interpolation formula. If it is required near the end of the table, we use NDFIF.

Eq. 1: Find the 1st & 2nd derivatives of \( f(x) \) at \( x = 1.5 \) for the given data:

\( x_i \): 1.5 2.0 3.5 5.0 7.0

\( y_i = f(x_i) \): 8.395 13.625 28.375 59

To find the difference table is given below:

<table>
<thead>
<tr>
<th>( x_i )</th>
<th>( y_i = f(x_i) )</th>
<th>( \Delta y )</th>
<th>( \Delta^2 y )</th>
<th>( \Delta^3 y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>8.395</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>13.625</td>
<td>5.230</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.5</td>
<td>28.375</td>
<td>14.750</td>
<td>0.95</td>
<td></td>
</tr>
<tr>
<td>5.0</td>
<td>59</td>
<td>30.675</td>
<td>4.5</td>
<td>0.95</td>
</tr>
<tr>
<td>7.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Here \( x = 1.5, x_0 = 1.5, h = 0.5 \), so \( \text{max } p = \frac{x - x_0}{h} = 0 \)

Using NDFIF, we get:

\[ y(x) = y_0 + \frac{p}{1!} \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \cdots \]

\[ y(x_0 + ph) = y_0 + p \Delta y_0 + \frac{(p)(p-1)}{2!} \Delta^2 y_0 + \frac{(p^2 - 3p + 2)}{6} \Delta^3 y_0 + \cdots \]

\[ y'((x_0 + ph)) = \frac{1}{h} \left[ \Delta y_0 + \frac{(p)(p-1)}{2} \Delta^2 y_0 + \frac{(3p^2 - 6p + 2)}{6} \Delta^3 y_0 + \cdots \right] \]

\[ y'((x_0 + ph)) = \frac{1}{0.5} \left[ \Delta y_0 + (\frac{1}{2})(p^2 - 3p + 2) \Delta^3 y_0 + \cdots \right] = \frac{\Delta y_0}{0.5} \]

\[ y'((1.5 + 0.5)) = \frac{\Delta y_0}{0.5} = \frac{5.230}{0.5} = 10.46 \]
(vi) 
\[ y'(1.5) = 4.75 \]
\[ \frac{dy}{dx} \text{ at } x = 1.5 \]
Again differentiating w.r.t \( p \), we get
\[ y''(x_0 + ph) = \frac{1}{h^2} \left[ \Delta^2 y_0 + (p - 1) \Delta^3 y_0 + \cdots \right] \]
\[ \text{i.e. } y''(x_0 + ph) = \frac{1}{(0.5)^2} \left[ 3 + (0 - 1)(0.75) \right] \]
\[ y''(1.5) = \frac{1}{h^2} \left[ \Delta^2 y_0 + \Delta^3 y_0 + \cdots \right] \]
\[ \left( \frac{d^2y}{dx^2} \right) \text{ at } x = 1.5 \]

(2) From the following table, find the values of \( \frac{dy}{dx}, \frac{d^2y}{dx^2} \) at \( x = 2.03 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>1.96</th>
<th>1.98</th>
<th>2.00</th>
<th>2.02</th>
<th>2.04</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0.7825</td>
<td>0.7739</td>
<td>0.7651</td>
<td>0.7563</td>
<td>0.7473</td>
</tr>
</tbody>
</table>

The difference table is as under:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( \Delta y )</th>
<th>( \Delta^2 y )</th>
<th>( \Delta^3 y )</th>
<th>( \Delta^4 y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.96</td>
<td>0.7825</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.98</td>
<td>0.7739</td>
<td>-0.0086</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.00</td>
<td>0.7651</td>
<td>-0.0088</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.02</td>
<td>0.7563</td>
<td>-0.0088</td>
<td>0</td>
<td>0.0002</td>
<td></td>
</tr>
<tr>
<td>2.04</td>
<td>0.7473</td>
<td>-0.0090</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Here \( x = 2.03, x_n = 2.04, h = 0.02 \) so that \( p = \frac{x - x_n}{h} = -0.5 \).

Using Newton's backward interpolation formula, we get
\[ y(x) = y_n + \frac{p}{1!} \Delta y_n + \frac{p(p + 1)}{2!} \Delta^2 y_n + \frac{p(p + 1)(p + 2)}{3!} \Delta^3 y_n + \cdots \]

i.e.
\[ y(x_n + ph) = \frac{y_n + p \Delta y_n + \frac{p(p + 1)}{2} \Delta^2 y_n + \frac{p(p + 1)(p + 2)}{6} \Delta^3 y_n + \cdots}{h} \]

diff. w.r.t \( p \), we get
\[ y'(x_n + ph) = \frac{0 + (1) \Delta y_n + 2 \frac{p + 1}{2} \Delta^2 y_n + (3p^2 + 6p + 2) \Delta^3 y_n}{h} + \cdots \]
\[ y'(x_n + \phi) = \frac{1}{h} \left[ \nabla^2 y_{n+1} + \frac{(\phi + 1)}{2} \nabla^3 y_{n+1} + \frac{(\phi + 1)}{6} \nabla^4 y_{n+1} \right] \]

\[ y'(2.03) = \frac{1}{0.002} \left[ (-0.009) + 0 - \frac{1}{24} \left( -0.00002 \right) \right] \]

\[ y'(2.03) = -0.4488 \]

Again differentiating w.r.t. \( p \), we get:

\[ y''(x_n + \phi) = \frac{1}{h} \left[ 0 + \frac{(\phi + 1)}{2} \nabla^2 y_{n+1} + \frac{(\phi + 1)}{6} \nabla^3 y_{n+1} + \frac{(\phi + 1)}{12} \nabla^4 y_{n+1} \right] \]

\[ y''(2.03) = \frac{1}{0.002} \left[ (-0.002) + (0.5)(-0.00002) + \frac{1}{24} (-0.00004) \right] \]

\[ y''(2.03) = -1.0414 \]

\[ \left( \frac{dy}{dx} \right)_{x=2.03} = -0.4488 \quad \text{and} \quad \left( \frac{d^2y}{dx^2} \right)_{x=2.03} = -1.0414 \]
The process of evaluating a definite integral from a set of tabulated values of the integrand \( y = f(x) \) is called numerical integration. This process when applied to a function of a single variable, is known as quadrature.

**Newton-Cote's Quadratic Formula:**

Let \( y = f(x) \) take the values \( y_0, y_1, \ldots, y_n \) corresponding to \( x_0, x_1, \ldots, x_n \) where \( x_i = x_0 + ih, \quad i = 1, 2, 3, \ldots, n \) then

\[
\int_{x_0}^{x_n} y \, dx = nh \left[y_0 + \frac{h}{2} \Delta y_0 + \frac{h(n-3)}{12} \Delta^2 y_0 + \frac{h(n-2)}{24} \Delta^3 y_0 + \cdots \right]
\]

From the above formula, we derive the following three important quadrature rules by taking \( n = 1, 2 \) & \( 3 \) respectively.

**1) Trapezoidal Rule:** \( \frac{1}{2} \)

\[
\int_{x_0}^{x_n} y \, dx = \frac{h}{2} \left[(y_0+y_n) + 2(y_1+y_2+\cdots+y_{n-2}+y_{n-1})\right]
\]

**2) Simpson's 1/3rd Rule:** \( \frac{1}{3} \)

\[
\int_{x_0}^{x_n} y \, dx = \frac{h}{3} \left[(y_0+y_n) + 2(y_1+y_4+\cdots+y_{n-2}) + 4(y_2+y_3+\cdots+y_{n-1})\right]
\]

Note: While applying Simpson's 1/3rd rule, divide the given interval into even no. of sub-intervals.

**3) Simpson's 3/8th Rule:** \( \frac{3}{8} \)

\[
\int_{x_0}^{x_n} y \, dx = \frac{3h}{8} \left[(y_0+y_n) + 2(y_1+y_6+\cdots+y_{n-6}+y_{n-3}) + 3(y_2+y_4+y_5+\cdots+y_{n-2}+y_{n-1})\right]
\]

Note: While applying Simpson's 3/8th rule, no. of sub-interval is taken as multiple of 3.

**Eq. 1.** Use trapezoidal rule to estimate the integral \( \int e^x \, dx \) using or taking 10 intervals.

Let \( y = e^x \) then \( x_0 = 0, \quad x_n = 2, \quad n = 10 \) so that \( h = \frac{x_n - x_0}{n} = \frac{2-0}{10} = 0.2 \).
Given:

\[ x : 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1.0 \quad 1.2 \quad 1.4 \quad 1.6 \quad 1.8 \quad 2.0 \]

\[ y : (y_0) \quad (y_1) \quad (y_2) \quad (y_3) \quad (y_4) \quad (y_5) \quad (y_6) \quad (y_7) \quad (y_8) \quad (y_9) \]

\[
\text{By Trapezoidal rule, we have}
\]

\[
\int_{0}^{2} y \, dx = \frac{h}{2} \left[ (y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1}) \right]
\]

\[
\Rightarrow \int_{0}^{2} \text{e}^x \, dx = \frac{(0.2)}{2} \left[ (1 + 0.533) + 2(1.0408 + 1.1935 + 1.4333 + 1.8983 + 1.983 + 1.2935 + 1.183) \right]
\]

\[ = 1.71901 \]

2. Use Simpson's 1/3rd rule to find \( \int_{0}^{2} \text{e}^x \, dx \) by taking 9 ordinates

So, let \( y = \text{e}^x \). Here \( x_0 = 0, x_n = 0.6, n = 6 \) so that \( h = \frac{x_n - x_0}{n} = 0.1 \)

\[ x : 0 \quad 0.1 \quad 0.2 \quad 0.3 \quad 0.4 \quad 0.5 \quad 0.6 \]

\[ y : 1 \quad 0.9900 \quad 0.9608 \quad 0.9389 \quad 0.9168 \quad 0.8949 \quad 1 \]

By Simpson's 1/3rd rule, we have

\[
\int_{0}^{2} y \, dx = \frac{h}{3} \left[ (y_0 + y_6) + 2(y_1 + y_3 + y_5) \right]
\]

\[
\Rightarrow \int_{0}^{2} \text{e}^x \, dx = \frac{(0.2)}{3} \left[ (1 + 0.6931) + 2(0.9608 + 0.9389 + 0.9168 + 0.8949) \right]
\]

\[ = 0.85351 \]

3. Complete the values of \( \int_{0}^{2} (\sin x - \log x + \text{e}^x) \, dx \) using Simpson's 3/8th rule

So, let \( y = \sin x - \log x + \text{e}^x \). Here \( x_0 = 0.2, x_n = 1.4, n = 6 \),

so that \( h = \frac{x_n - x_0}{n} = 0.2 \)

\[ x : 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1.0 \quad 1.2 \quad 1.4 \]

\[ y : 3.0295 \quad 3.7985 \quad 3.9968 \quad 3.1660 \quad 3.5398 \quad 4.0293 \quad 4.7042 \]

By Simpson's 3/8th rule, we have

\[
\int_{0}^{2} y \, dx = \frac{h}{8} \left[ (y_0 + y_6) + 2(y_1 + y_3 + y_5) \right]
\]

\[
\Rightarrow \int_{0}^{2} (\sin x - \log x + \text{e}^x) \, dx = \frac{0.2}{8} \left[ (3.0295 + 4.7042) + 2(3.1660 + 3.5398 + 4.0293) \right]
\]

\[ = 4.0530 \]
4. Evaluate \( \int_0^6 \frac{dx}{1+x^2} \) by using i) trapezoidal rule

\[ \int_0^6 \frac{dx}{1+x^2} = \frac{1}{6} \left[ \frac{x}{1+x^2} \right]_0^6 = 1.4056 \]

ii) Simpson's 1/3 rule & iii) Simpson's 3/8 rule. Compare the result with its actual value.

Let \( y = \frac{1}{1+x^2} \), then \( x_0 = 0, x_n = 6, n = 6 \), so that \( \Delta x = \frac{x_n - x_0}{n} = 1 \)

\( x : 0, 1, 2, 3, 4, 5, 6 \)
\( y : 1, 0.5, 0.2, 0.1, 0.0588, 0.0335, 0.0247 \)
\( (y_0) (y_1) (y_2) (y_3) (y_4) (y_5) \)

i) By trapezoidal rule, we have

\[ \int_0^6 \frac{1}{1+x^2} \, dx = \frac{1}{2} \left[ (y_0+y_6) + 2(y_1+y_5) + 4(y_2+y_4) + 6(y_3) \right] \]

\[ = \frac{1}{2} \left[ (1+0.0247) + 2(0.5+0.2+0.1+0.0588+0.0335) \right] \]

\[ = 1.410 \] 

ii) By Simpson's 1/3 rule, we have

\[ \int_0^6 \frac{1}{1+x^2} \, dx = \frac{1}{3} \left[ (y_0+y_6) + 2(y_1+y_5) + 4(y_2+y_4) \right] \]

\[ = \frac{1}{3} \left[ (1+0.0247) + 2(0.5+0.2+0.1+0.0588+0.0335) \right] \]

\[ = 1.3662 \]

iii) By Simpson's 3/8 rule, we have

\[ \int_0^6 \frac{1}{1+x^2} \, dx = \frac{3}{8} \left[ (y_0+y_6) + 2(y_1+y_5) + 3(y_2+y_4+y_3) \right] \]

\[ = \frac{3}{8} \left[ (1+0.0247) + 2(0.5+0.2+0.1+0.0588+0.0335) \right] \]

\[ = 1.3571 \]

By actual integration, we have

\[ \int_0^6 \tan^{-1}(x) \, dx = \left[ \tan^{-1}(x) \right]_0^6 = \tan^{-1}(6) - \tan^{-1}(0) \]

This shows that the value of the integral found by trapezoidal rule is nearest to the actual value.
2. Curve Fitting

*Curve fitting is the method of finding the eqn of a curve that approximates the given set of points.*

*Method of least squares:*

Among many methods available for curve fitting, the most popular method is the method of least squares.

1. Fitting in a straight line:

Let \( y = a + bx \) \((a, b \text{ are parameters})\) be the straight line to be fitted to the given set of \( n \) points \((x_i, y_i), i = 1, 2, \ldots, n\).

The normal equations are:

\[
\begin{align*}
na + b \sum x_i &= \sum y_i; \\
\sum x_i + b \sum x_i^2 &= \sum x_i y_i; \\
\sum x_i^2 + b \sum x_i^3 &= \sum x_i^2 y_i; \\
\sum x_i^3 + b \sum x_i^4 &= \sum x_i^3 y_i.
\end{align*}
\]

The parameters \( a \) and \( b \) are obtained by solving the normal equations. Substituting \( a, b \) in eq. (1) we get the req. straight line of best fit.

2. Fitting a Parabola (second degree polynomial):

Let \( y = a + bx + cx^2 \) \((a, b, c \text{ are parameters})\) be the parabola to be fitted to the given set of \( n \) points \((x_i, y_i), i = 1, 2, \ldots, n\).

The normal eqns. are:

\[
\begin{align*}
na + b \sum x_i + c \sum x_i^2 &= \sum y_i; \\
\sum x_i + b \sum x_i^2 + c \sum x_i^3 &= \sum x_i y_i; \\
\sum x_i^2 + b \sum x_i^3 + c \sum x_i^4 &= \sum x_i^2 y_i; \\
\sum x_i^3 + b \sum x_i^4 + c \sum x_i^5 &= \sum x_i^3 y_i.
\end{align*}
\]

Solve the normal eqns. for \( a, b, c \) and substitute them in eq. (1), we get the req. parabola of best fit.

eg 1: By the method of least squares, find the straight line that best fits following data and estimate the value of \( y \).

Corresponding to \( x = 4 \):

\[
\begin{align*}
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
4 & 6 & 8 & 10
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{cccc}
4 & 6 & 8 & 10
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{cccc}
14 & 26 & 40 & 55
\end{array}
\end{align*}
\]

Let \( y = a + bx \) \((a, b \text{ are parameters})\) be the straight line to be fitted to the given data.
The normal equations are \[ na + b \sum x_i = \sum y_i \]
\[ a \sum x_i + b \sum x_i^2 = \sum x_i y_i \]

\[
\begin{array}{cccc}
    x_i & y_i & x_i^2 & x_i y_i \\
    1 & 14 & 1 & 14 \\
    2 & 25 & 4 & 50 \\
    3 & 40 & 9 & 120 \\
    4 & 55 & 16 & 880 \\
    5 & 65 & 25 & 1280 \\
\end{array}
\]

\[ \sum x_i = 15 \quad \sum y_i = 204 \quad \sum x_i^2 = 55 \quad \sum x_i y_i = 748 \]

Substituting these values in normal equations, we get

\[ 6a + 15b = 204 \]
\[ 15a + 55b = 748 \]

Solving for \( a, b \) we obtain \( a = 0, b = 13.6 \)

Substituting \( a, b \) values in (1), we get \[ y = 13.6x \]

\( \therefore \) which is a straight line.

Also when \( x = 7 \), then \( y = 13.6 \times 7 = 95.2 \)

2. If \( p \) is the pull required to lift a load \( w \) by means of a pulley block, find a linear law of the form \( p = mw + c \), connecting \( p \) with \( w \) using following data.

\( p \): 12, 15, 20, 25 where \( p \) and \( w \) are taken \( kg \) and \( m\) respectively.

\( w \): 50, 60, 100, 180 when \( w = 150 \ kg \) \( \therefore \) \( (1 \ kg = 9.8 \ N) \)

\[ a = \frac{b}{c} \]

Given \( p = c + mw \), where \( c, m \) are parameters.

The corresponding normal equations are \[ n c + m \sum w = \sum p \]

Here \( n = 4 \).

<table>
<thead>
<tr>
<th>( w )</th>
<th>( p )</th>
<th>( w^2 )</th>
<th>( wp )</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>12</td>
<td>2500</td>
<td>600</td>
</tr>
<tr>
<td>40</td>
<td>15</td>
<td>1600</td>
<td>600</td>
</tr>
<tr>
<td>100</td>
<td>21</td>
<td>10000</td>
<td>2100</td>
</tr>
<tr>
<td>180</td>
<td>25</td>
<td>14400</td>
<td>4500</td>
</tr>
</tbody>
</table>

\[ \sum w = 340 \quad \sum p = 78 \quad \sum w^2 = 51800 \quad \sum wp = 6750 \]
Substituting these values in normal equations, we get

\[ 4c + 340m = 73 \]
\[ 340c + 31800m = 6375 \]

Solving for \( c, m \), we obtain \( c = 8.2959, m = 0.1859 \)

Hence the line of best fit is

\[ p = 0.1859w + 8.2959 \]

When \( w = 150 \) kg - wt \( \Rightarrow \) \( p = 80.4609 \) kg - wt

3. The temperature \( T(\text{°C}) \) and length \( L(\text{mm}) \) of a heated rod are given below. If \( L = a_0 + a_1T \), find the best values of \( a_0, a_1 \)

<table>
<thead>
<tr>
<th>( T )</th>
<th>( L )</th>
<th>( T \times L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>800.3</td>
<td>160060.00</td>
</tr>
<tr>
<td>30</td>
<td>800.4</td>
<td>240120.00</td>
</tr>
<tr>
<td>40</td>
<td>800.6</td>
<td>320240.00</td>
</tr>
<tr>
<td>50</td>
<td>800.7</td>
<td>400350.00</td>
</tr>
<tr>
<td>60</td>
<td>800.9</td>
<td>480480.00</td>
</tr>
<tr>
<td>70</td>
<td>801.0</td>
<td>560600.00</td>
</tr>
</tbody>
</table>

\[ \sum T = 270 \]
\[ \sum L = 48010000 \]
\[ \sum T^2 = 13900000 \]
\[ \sum TL = 2162010000 \]

Substituting these values in normal equations, we get

\[ 6a_0 + 340a_1 = 4803 \]
\[ 3400a_0 + 139000a_1 = 816201 \]

Solving both we get \( a_0 = 799.9945, a_1 = 0.0146 \)
UNIT-V

NUMERICAL SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS
In this chapter we find the numerical solutions of ordinary differential equations using the following methods:

a) Single-step methods
   i) Taylor's series method
   ii) Euler's method
   iii) Modified Euler's method
   iv) Runge-Kutta method of 4th order
b) Multi-step methods
   i) Adams-Bashforth-Moulton Predictor-Corrector method

1. Taylor's series method:
   Consider the 1st order equation \( \frac{dy}{dx} = f(x, y) \) — (1) with the initial condition \( y(x_0) = y_0 \).

   If (1) w.r.t. \( x \) successively & substitute \( x_0, y_0 \) values, we can get \( y_0', \ y_0'', \ y_0''', \ldots \).

   Taylor series expansion of \( y(x) \) about \( x = x_0 \) is given by
   \[
   y(x) = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \cdots \] — (2)

   Put \( x = x_1 \) in (2), we get
   \[
   y(x_1) = y_0 + \frac{(x_1-x_0)}{1!} y_0' + \frac{(x_1-x_0)^2}{2!} y_0'' + \frac{(x_1-x_0)^3}{3!} y_0''' + \cdots \]

   If we can find \( y(x_2), y(x_3), \ldots, y(x_n) \)

   Example:
   1. Find by Taylor's method the value of \( y \) at \( x = 0.1 \) to 5 places of decimals from \( \frac{dy}{dx} = x^2y - 1 \), \( y(0) = 1 \)

   Set:
   Given \( y' = x^2y - 1 \) — (1), \( x_0 = 0 \), \( y_0 = 1 \)

   \[ y_0' = x_0^2y_0 - 1 = -1 \]

   Substitute (1) successively & substitute the values of \( x_0, y_0 \), we get
   \[
   y'' = x^2y + 2xy \quad y_0'' = x_0^2y_0' + 2x_0y_0 = 0 \]
   \[
   y''' = x^2y'' + 4xy' + 2y \quad y_0''' = x_0^2y_0'' + 4x_0y_0' + 2y_0 = 2 \]
   \[
   y'''' = x^2y''' + 6xy'' + 6y' \quad y_0'''' = x_0^2y_0''' + 6x_0y_0'' + 6y_0' = -6
   

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By Taylor's series method, we have

\[ y(x) = y_0 + \frac{(x-x_0)}{1!} y'_0 + \frac{(x-x_0)^2}{2!} y''_0 + \frac{(x-x_0)^3}{3!} y'''_0 + \frac{(x-x_0)^4}{4!} y''''_0 + \ldots \]  

(2)

Substituting the values of \( x_0, y_0, y'_0, y''_0, y'''_0, \ldots \) in (2), we obtain

\[ y(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{4} + \ldots \]  

(3)

Put \( x = 0.1 \) & \( x = 0.2 \). In (3), we get

\[ y(0.1) = 1 - (0.1) + \frac{(0.1)^2}{3} - \frac{(0.1)^4}{4} + \ldots = 0.90031 \]

\[ y(0.2) = 1 - (0.2) + \frac{(0.2)^2}{3} - \frac{(0.2)^4}{4} + \ldots = 0.80227 \]

2. Employee Taylor's method to obtain approximate value of \( y \) at \( x = 0.2 \) for the ODE \( \frac{dy}{dx} = 2y + 3e^x \), \( y(0) = 0 \). Compare the numerical solution obtained with the exact solution.

Equation:

\[ y' = 2y + 3e^x \quad y(0) = 0 \Rightarrow x_0 = 0, \ y_0 = 0 \]

Differentiate successively & substitute the values of \( x_0 = 0, y_0, y'_0, \ldots \), we get

\[ y_0' = \frac{2y_0 + 3e^x_0}{3} = 3 \]

\[ y'' = 2y'' + 3e^x \]

\[ y''_0 = 2y''_0 + 3e^{x_0} = q \]

\[ y''' = 2y''' + 3e^x \]

\[ y'''_0 = 2y'''_0 + 3e^{x_0} = 21 \]

\[ y'''' = 2y'''' + 3e^x \]

\[ y''''_0 = 2y''''_0 + 3e^{x_0} = 45 \]

By Taylor's series method, we have

\[ y(x) = y_0 + \frac{(x-x_0)}{1!} y'_0 + \frac{(x-x_0)^2}{2!} y''_0 + \frac{(x-x_0)^3}{3!} y'''_0 + \frac{(x-x_0)^4}{4!} y''''_0 + \ldots \]  

(2)

Substituting the values of \( x_0, y_0, y'_0, y''_0, y'''_0, \ldots \) in (2), we get

\[ y(x) = 3x + \frac{q}{2} x^2 + \frac{q}{8} x^3 + \frac{15}{8} x^4 + \ldots \]

Put \( x = 0.2 \), we get

\[ y(0.2) = 3(0.2) + \frac{q}{2} (0.2)^2 + \frac{q}{8} (0.2)^3 + \frac{15}{8} (0.2)^4 + \ldots \]

To find the exact value is

\[ = 0.8110 \]  

(3)

Given eq'n can be written as \( \frac{dy}{dx} - 2y = 3e^x \) which is linear.

\[ \text{I.F.} = e \int \frac{1}{p} \, dx = e \int \frac{1}{2} \, dx = e^{-2x} \]
The general sol is \( y(e^{2x}) = \int 3e^x(e^{-2x}) \, dx + c \)

\[ y e^{-2x} = -3e^{-x} + c \quad \text{(i)} \quad y(x) = -3e^x + ce^{2x} \]

Put \( x = 0 \), we get \( y(0) = c \cdot e^0 - 3e^0 \)

\[ = 0 = c - 3 \Rightarrow c = 3 \quad (\because y(0) = 0) \]

Thus the exact sol is \( y(x) = 3(e^{2x} - e^x) \quad (4) \)

Put \( x = 0.2 \) in (4), we get \( y(0.2) = 3(e^{0.4} - e^{0.2}) \)

\[ = 0.8113 \quad (5) \]

Compare (3) & (5) it is clear that 3 approximates to the exact value up to 3 decimals.

3. Solve by Taylor's series method the eqn \( \frac{dy}{dx} = \log(xy) \) for \( y(1.1) = y(1.2) \), given \( y(1) = 2 \).

Set: \( \frac{dy}{dx} = \log(xy) \)

\[ \Rightarrow y' = \log_x + \log_y \quad (1) \quad x_o = 1, y_o = 2 \]

\[ \Rightarrow y'_0 = \log_x + \log_2 = \log_1 + \log_2 = 0.6931 \]

Diff (1) successively \& substituting the values of \( x_0, y_0 \)

we get \( y'' = \frac{1}{x} + \frac{1}{y} \cdot y' \quad \Rightarrow \quad y''_0 = \frac{1}{1} + \frac{1}{2} \cdot y'_0 = 1.3466 \)

\[ y''' = \frac{-1}{x^2} + \left( \frac{y'' \cdot y' - y''_0 \cdot y'}{y'} \right) \quad \Rightarrow \quad y'''_0 = -0.9468 \]

By Taylor's series expansion, we get

\[ y(x) = y_0 + (x-x_0) \frac{y'_0}{1!} + \frac{(x-x_0)^2}{2!} \frac{y''_0}{2!} + \frac{(x-x_0)^3}{3!} \frac{y'''_0}{3!} + \cdots \quad (2) \]

Substituting the values of \( x_0, y_0, y'_0, y''_0, y'''_0 \), we get

\[ y(x) = 2 + (x-1)(0.6931) + \frac{(x-1)^2}{2!}(1.3466) + \frac{(x-1)^3}{3!}(-0.9468) + \cdots \]

Put \( x = 1.1, 1.2 \), we get

\[ y(1.1) = 2 + (0.1)(0.6931) + \frac{(0.1)^2(1.3466)}{2} + \frac{(0.1)^3(-0.9468)}{6} \]

\[ y(1.2) = 2.0760 \]
Euler’s Method:

General formula for Euler’s method is $y_{n+1} = y_n + h \cdot f(x_n, y_n)$.

Put $n=1$ to get $y(x_1) = y_0 + h \cdot f(x_0, y_0)$

Put $n=2$ to get $y(x_2) = y_1 + h \cdot f(x_1, y_1)$

1. Solve by Euler’s method the equation $\frac{dy}{dx} = x + y$, $y(0) = 0$

   Choose $h = 0.2$ and conclude $y(0.4)$ & $y(0.6)$

   $f(x, y) = x + y$; $y(0) = 0 \Rightarrow x_0 = 0, y_0 = 0$, $h = 0.2$

   Euler’s formula is $y(x_n) = y_{n-1} + h \cdot f(x_{n-1}, y_{n-1})$

   Put $n=1$; $y(x_1) = y_0 + h \cdot f(x_0, y_0)$
   
   $y(0.2) = 0 + 0.2 \cdot f(0, 0) = 0$

   Put $n=2$; $y(x_2) = y_1 + h \cdot f(x_1, y_1)$
   
   $y(0.4) = 0 + (0.2) \cdot f(0.2, 0) = 0.04$

   Put $n=3$; $y(x_3) = y_2 + h \cdot f(x_2, y_2)$
   
   $y(0.6) = 0.04 + (0.2) \cdot f(0.4, 0.04) = 0.1280$

   $\therefore y(0.4) = 0.04$, $y(0.6) = 0.1280$
Modified Euler's Method:

Consider the 1st order ODE \( \frac{dy}{dx} = f(x, y) \) with the initial condition \( y(x_0) = y_0 \) then 1st approximate modified Euler's formula is

\[
y^{(1)} = y_0 + \frac{h}{2} \left[ f(x_0, y_0) + f(x_1, y_1^{(0)}) \right]
\]

\[
y_1^{(1)} = y_0 + \frac{h}{2} \left[ f(x_0, y_0) + f(x_1, y_0^{(0)}) \right]
\]

\[
y_1^{(2)} = y_0 + \frac{h}{2} \left[ f(x_0, y_0) + f(x_1, y_1^{(1)}) \right]
\]

\[
y_2^{(0)} = y_0 + \frac{h}{2} \left[ f(x_0, y_0) + f(x_1, y_1^{(1)}) \right]
\]

Continue this procedure till to get 2 successive approximations are equal.

For 2nd approximation use \( y^{(2)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_2, y_2^{(0)}) \right] \)

\[
y_2^{(1)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_2, y_2^{(0)}) \right]
\]

\[
y_2^{(2)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_2, y_2^{(1)}) \right]
\]

\[
y_2^{(3)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_2, y_2^{(1)}) \right]
\]

For 3rd approximation we use \( y^{(3)} = y_2 + \frac{h}{2} \left[ f(x_2, y_2) + f(x_3, y_3^{(0)}) \right] \)

\[
y_3^{(1)} = y_2 + \frac{h}{2} \left[ f(x_2, y_2) + f(x_3, y_3^{(0)}) \right]
\]

\[
y_3^{(2)} = y_2 + \frac{h}{2} \left[ f(x_2, y_2) + f(x_3, y_3^{(1)}) \right]
\]

\[
y_3^{(3)} = y_2 + \frac{h}{2} \left[ f(x_2, y_2) + f(x_3, y_3^{(1)}) \right]
\]
Given \( \frac{dy}{dx} = \frac{y-x}{y+x} \) with boundary condition \( y = 1 \) when \( x = 0 \); find the approximately \( y \) for \( x = 0.1 \) by Euler's modified method.

Here \( f(x, y) = \frac{y-x}{y+x} \), \( x_0 = 0 \), \( y_0 = 1 \) and \( h = \frac{x-x_0}{h} = \frac{0.1}{0.02} = 5 \).

To find \( y_1 \) i.e. \( y(0.02) \):

\[ x_1 = x_0 + h = 0 + 0.02 = 0.02 \]
\[ f(x_0, y_0) = \frac{y_0-x_0}{y_0+x_0} = \frac{1-0}{1+0} = 0 \]

By Euler's formula, \( y_1^{(0)} = y_0 + h \cdot f(x_0, y_0) = 1 + 0.02(0) = 1.02 \)

By Euler's modified formula, \( y_1^{(1)} = y_0 + h/2 \left[ f(x_0, y_0) + f(x_1, y_1^{(0)}) \right] = 1.0196 \)

For \( n = 2 \):
\[ y_2^{(1)} = y_0 + h/2 \left[ f(x_0, y_0) + f(x_1, y_1^{(0)}) \right] = 1.0196 \]

Since \( y_1^{(1)} = y_1^{(0)} \); we get \( y_1 = y(0.02) = 1.0196 \)

To find \( y_2 \) i.e. \( y(0.04) \):

\[ x_2 = x_0 + h = 0.02 + 0.02 = 0.04 \]
\[ f(x_1, y_1) = \frac{y_1-x_1}{y_1+x_1} = 0.9685 \]

By Euler's formula, \( y_2^{(0)} = y_1 + h \cdot f(x_1, y_1) = 1.0388 \)

By modified Euler's formula, \( y_2^{(1)} = y_1 + h/2 \left[ f(x_1, y_1) + f(x_2, y_2^{(0)}) \right] = 1.0385 \)

For \( n = 2 \):
\[ y_2^{(2)} = y_1 + h/2 \left[ f(x_1, y_1) + f(x_2, y_2^{(1)}) \right] = 1.0385 \]

Since \( y_2^{(1)} = y_2^{(0)} \); \( y_2 = y(0.04) = 1.0385 \)

To find \( y_3 \) i.e. \( y(0.06) \):

To find \( x_3 = x_2 + h = 0.04 + 0.02 = 0.06 \)
\[ f(x_2, y_2) = \frac{y_2-x_2}{y_2+x_2} = 0.9258 \]

By Euler's formula we get \( y_3^{(0)} = y_2 + h \cdot f(x_2, y_2) = 1.0559 \)
By modified Euler's method, \( y_3^{(n)} = y_3 + h \left[ f(x_3, y_3) + f(x_3, y_3^{(n)}) \right] \)

For \( n = 1 \):
\[
y_3^{(1)} = y_3 + h \left[ f(x_3, y_3) + f(x_3, y_3) \right] = 1.0569
\]

For \( n = 2 \):
\[
y_3^{(2)} = y_3 + \frac{h}{2} \left[ f(x_3, y_3) + f(x_3, y_3^{(1)}) \right] = 1.05693
\]

Since \( y_3^{(1)} \approx y_3^{(2)} \), \( y_3 = 1.0569 = y(0.06) \)

To find \( y_4 \), \( y(0.08) \):

\[\alpha_4 = x_3 + h = 0.06 + 0.02 = 0.08, \quad x_3 = 0.06, \quad y_3 = 1.0569, \quad h = 0.02\]

\[
f(x_3, y_3) = \frac{y_3 - x_3}{y_3 + x_3} = 0.8925
\]

By Euler's method, we have \( y_4^{(n)} = y_3 + h \left[ f(x_3, y_3) \right] = 1.0546 \)

By Euler's modified method, \( y_4^{(n)} = y_3 + \frac{h}{2} \left[ f(x_3, y_3) + f(x_3, y_3^{(n)}) \right] \)

For \( n = 1 \):
\[
y_4^{(1)} = y_3 + \frac{h}{2} \left[ f(x_3, y_3) + f(x_3, y_3) \right] = 1.0342
\]

For \( n = 2 \):
\[
y_4^{(2)} = y_3 + \frac{h}{2} \left[ f(x_3, y_3) + f(x_3, y_3^{(1)}) \right] = 1.0342
\]

\[y_4^{(n)} \approx y_4^{(2)}, \quad y_4 = y(0.08) = 1.0342\]

To find \( y_5 \), \( y(0.1) \):

\[\alpha_5 = 0.08, \quad y_4 = 1.0342, \quad \alpha_5 = x_4 + h = 0.08 + 0.02 = 0.1\]

\[
f(x_4, y_4) = \frac{y_4 - x_4}{y_4 + x_4} = 0.8614
\]

By Euler's method, \( y_5^{(n)} = y_4 + h \cdot f(x_4, y_4) = 1.0914 \)

By modified Euler's method, \( y_5^{(n)} = y_4 + \frac{h}{2} \left[ f(x_4, y_4) + f(x_4, y_4^{(n)}) \right] \)

For \( n = 1 \):
\[
y_5^{(1)} = y_4 + \frac{h}{2} \left[ f(x_4, y_4) + f(x_4, y_4) \right] = 1.091
\]

For \( n = 2 \):
\[
y_5^{(2)} = y_4 + \frac{h}{2} \left[ f(x_4, y_4) + f(x_4, y_4^{(1)}) \right] = 1.0911
\]

\[y_5^{(n)} \approx y_5^{(2)}, \quad y_5 = 1.0911 = y(0.1)\]
Runge-Kutta method of fourth order:

Consider \( \frac{dy}{dx} = f(x, y) \) with \( y(x_0) = y_0 \)

To find \( y_1 \) i.e. \( y(x_1) \):

\[
egin{align*}
K_1 &= h \cdot f(x_0, y_0) \\
K_2 &= h \cdot f(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}) \\
K_3 &= h \cdot f(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}) \\
K_4 &= h \cdot f(x_0 + h, y_0 + K_3) \\
\end{align*}
\]

To find \( y_2 \) i.e. \( y(x_2) \):

\[
egin{align*}
K_{21} &= h \cdot f(x_1, y_1) \\
K_2 &= h \cdot f(x_1 + \frac{h}{2}, y_1 + \frac{K_{21}}{2}) \\
K_3 &= h \cdot f(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}) \\
K_4 &= h \cdot f(x_1 + h, y_1 + K_3) \\
\end{align*}
\]

In general,

\[
y_{y+1} = y_{y+1} + \frac{1}{6} \left(K_1 + 2K_2 + 2K_3 + K_4\right)
\]

where \( K_1 = h \cdot f(x_{y+1}, y_{y+1}) \)

\[
K_2 = h \cdot f(x_{y+1} + \frac{h}{2}, y_{y+1} + \frac{K_1}{2}) \\
K_3 = h \cdot f(x_{y+1} + \frac{h}{2}, y_{y+1} + \frac{K_2}{2}) \\
K_4 = h \cdot f(x_{y+1} + h, y_{y+1} + K_3)
\]

Examples:

1. Apply Runge-Kutta method of fourth order to find an approximate value of \( y \) when \( x = 0.2 \), given that \( \frac{dy}{dx} = x + y \) and \( y = 1 \) when \( x = 0 \).

Solve here \( f(x, y) = x + y \), \( x_0 = 0 \), \( y_0 = 1 \), \( h = 0.2 \)

\[x_1 = x_0 + h = 0 + 0.2 = 0.2 \]

By Runge-Kutta method of fourth order, we have

\[
y_1 = y(x_1) = y_0 + \frac{1}{6} \left(K_1 + 2K_2 + 2K_3 + K_4\right)
\]

where \( K_1 = h \cdot f(x_0, y_0) \)

\[
K_2 = h \cdot f(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}) = (0.2) \cdot f(0, 1) = 0.2 \]

\[
K_3 = h \cdot f(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}) = (0.2) \cdot f(0.1, 1.1) = 0.24
\]

\[
; \text{i.e., } K_3 = 0.2 \cdot (0.1 + 1.1) = 0.24
\]
\[ k_3 = h \cdot f \left( x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2} \right) \]
\[ = (0.2) \cdot f \left( 0 + \frac{0.2}{2}, 1 + \frac{0.249}{2} \right) \]
\[ = (0.2) \cdot f (0.1, 1.12) \]
\[ = (0.2) \cdot f (0.1, 1.12) = 0.249 \]

Sub \( k_1, k_2, k_3, k_4 \) in (1)

\[ y_1 = y_0 + \frac{1}{6} \left[ k_1 + 2k_2 + 2k_3 + k_4 \right] = 1.2428 \]

2. Apply Runge-Kutta method to find an approximate value of \( y \)

for \( x = 0.2 \) in steps of 0.1, if \( \frac{dy}{dx} = x + y^2 \), given that \( y = 1 \) when \( x = 0 \).

Set \( f(x, y) = x + y^2 \), \( x_0 = 0 \), \( h = 0.1 \), \( y_0 = 1 \), \( x_1 = x_0 + h = 0.1 \)

To find \( y_1 \) i.e. \( y(0.1) \).

By Runge-Kutta method of fourth order, we have

\[ y_1 = y_0 + \frac{1}{6} \left[ k_1 + 2k_2 + 2k_3 + k_4 \right] \]  

where

\[ k_1 = h \cdot f(x_0, y_0) \]
\[ = (0.1) \cdot f(0, 1) \]
\[ = 0.1 \cdot (0 + 1^2) = 0.1 \]

\[ k_2 = h \cdot f \left( x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2} \right) \]
\[ = (0.1) \cdot f \left( 0 + \frac{0.1}{2}, 1 + \frac{0.1}{2} \right) \]
\[ = (0.1) \cdot f (0.05, 1.05) \]
\[ = (0.1) \cdot f (0.05, 1.05) = 0.1153 \]

\[ k_3 = h \cdot f \left( x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2} \right) \]
\[ = (0.1) \cdot f \left( 0 + \frac{0.1}{2}, 1 + \frac{0.1153}{2} \right) \]
\[ = (0.1) \cdot f (0.05, 1.0527) \]
\[ = (0.1) \cdot f (0.05, 1.0527) \]
\[ = 0.1169 \]

Sub \( k_1, k_2, k_3, k_4 \) in (1)

\[ y_1 = y_0 + \frac{1}{6} \left[ k_1 + 2k_2 + 2k_3 + k_4 \right] = 1.165 \]

To find \( y_2 \) i.e. \( y(0.2) \).

By Runge-Kutta method of fourth order, we have

\[ y_2 = y_1 + \frac{1}{6} \left[ k_1 + 2k_2 + 2k_3 + k_4 \right] \]
Where \( k_1 = h \cdot f (x_1, y_1) \)
\[ k_1 = (0.1) f (0.1, 1.1165) \]
\[ k_1 = (0.01) \left( 0.1 + 1.1165^2 \right) \]
\[ k_1 = 0.1143 \]

\[ k_b = h \cdot f \left( x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2} \right) \]
\[ k_b = (0.1) f \left( 0.1 + 0.1 \cdot \frac{1}{2}, 1.1165 + \frac{0.1143}{2} \right) \]
\[ k_b = (0.01) f (0.15, 1.1941) \]
\[ k_b = (0.01) \left( 0.15 + 1.1941^2 \right) = 0.1536 \]

Sub \( k_1, k_2, k_3, k_4 \) values in (2) we get

\[ y_2 = 1.1165 + \frac{1}{6} \left[ 0.1343 + 2 \left( 0.1343 + 0.1536 \right) + 0.1523 \right] \]
\[ y_2 = 1.2726 \]

Hence \( y(0.2) = 1.2726 \)
1. **ADAM'S - BASHFORTH - MOULTON PREDICTOR - CORRECTOR METHOD (ABM method)**

Consider \( \frac{dy}{dx} = f(x, y) \) with \( y(x_0) = y_0 \)

1. Calculate \( y_1, y_2, y_3 \), using the method's discussed in the previous section, i.e., Taylor's/modified euler's/Runge-Kutta method, if not given.
2. Find \( f_0 = f(x_0, y_0), f_1 = f(x_1, y_1), f_2 = f(x_2, y_2), \) or \( f_3 = f(x_3, y_3) \)
3. By **ADAM'S - BASHFORTH** predictor formula:
   \[ y_4^{(p)} = y_3 + \frac{h}{24} [55f_3 - 59f_2 + 37f_1 - 9f_0] \]
4. Calculate \( f_4 = f(x_4, y_4^{(p)}) \)
5. By **ADAM'S - BASHFORTH** corrector formula:
   \[ y_4^{(c)} = y_3 + \frac{h}{24} [9f_4 + 19f_3 - 5f_2 + f_1] \]

Again substitute this \( f_4 \) value in eq. (2) to obtain \( y_4^{(c)} \). Continue the above procedure until two successive corrector values \( y_4^{(c)} \) are equal.

6. Starting with \( y_1, y_2, y_3, y_4 \), we can obtain \( y_5 \) value using the following formula:
   \[ y_5^{(p)} = y_4 + \frac{h}{24} [55f_4 - 59f_3 + 37f_2 - 9f_1] \]
   \[ y_5^{(c)} = y_4 + \frac{h}{24} [9f_5 + 19f_4 - 5f_3 + f_2] \]

**Example:**

1. Given \( \frac{dy}{dx} = x^2(1+y) \) at \( y(1) = 1 \), \( y(1+\frac{1}{2}) \)

Given \( y_0 = 1, x_0 = 1, f(x_1, y_1) = x^2(1+y) \), \( h = 0.1 \)

\( x_1 = 1, x_2 = 1.2, x_3 = 1.4, x_4 = 1.6, x_5 = 1.8 \)

\( y_1 = 1 + 0.232, y_2 = 1.548, y_3 = 1.939 \)

To find \( y_4 \) i.e. \( y(1.4) \):

\( x_4 = x_3 + h = 1.4 \)

\( f_0 = f(x_0, y_0) = x_0^2(1+y_0) = 1.232 \)

\( f_1 = f(x_1, y_1) = x_1^2(1+y_1) = 1.61 \)

\( f_2 = f(x_2, y_2) = (x_2)^2(1+y_2) = 3.669 \)

\( f_3 = f(x_3, y_3) = (x_3)^2(1+y_3) = 8.019 \)

\( f_4 = f(x_4, y_4^{(p)}) \)

\( y_4^{(p)} = y_3 + \frac{h}{24} [55f_3 - 59f_2 + 37f_1 - 9f_0] \)

\( y_4^{(c)} = y_3 + \frac{h}{24} [9f_4 + 19f_3 - 5f_2 + f_1] \)

\( y_4 = y_4^{(p)} \) or \( y_4 = y_4^{(c)} \)
By Adam's - Bashforth predictor formula \( y_q(t) = y_1 + \frac{b}{24} \left[ 55y_3 - 57y_6 + 29y_7 - y_8 \right] \)

= 1.939 + \frac{0.1}{24} \left[ 55(3.667) - 57(2.701) + 29(6.035) - (2.70) \right]

= 2.535

Now \( f_q = f(x_q, y_q(t)) = f(1.4, 2.535) = 7.007 \)

By Adam's - Multon corrector formula \( y_q(t) = y_3 + \frac{h}{24} \left[ 9f_q + 19f_3 - 5f_1 - f_7 \right] \)

= 1.939 + \frac{0.1}{24} \left[ 9(7.007) + 19(3.667) - 5(2.701) - (2.70) \right]

\( y_q = 8.535 \)

Since \( y_q = y_q(t) \), \( y_q = 8.535 \)

Hence \( y(1.4) = 2.535 \)

\[ eq.(24) \]

2. Using Adam's - Bashforth method, find \( y(4.4) \) given \( 5x^2 + y^2 = 2 \), \( y(4) = 1 \), \( y(4.1) = 1.0049 \), \( y(4.2) = 1.0097 \) and \( y(4.3) = 1.0143 \).

Set \( f(x, y) \) can be written as \( \frac{dy}{dx} = 2 - y^2 \) \( \Rightarrow y' = \frac{2 - y^2}{5x} \)

Here \( f(x, y) = \frac{2 - y^2}{5x} \), \( x_0 = 4 \), \( y_0 = 1 \), \( x_1 = 4.1 \), \( x_2 = 4.2 \), \( x_3 = 4.3 \), \( x_4 = 4.4 \), \( h = 0.1 \)

To find \( y_4 \) i.e., \( y(4.4) \)

\( x_4 = x_3 + h = 4.3 + 0.1 = 4.4 \)

\( f_0 = f(x_0, y_0) = \frac{2 - 1}{5(4)} = \frac{1}{20} = 0.05 \)

\( f_3 = f(x_3, y_3) = \frac{2 - 1.0049^2}{5(4.3)} = 0.0469 \)

\( f_1 = f(x_1, y_1) = \frac{2 - 1.0097^2}{5(4.1)} = 0.0483 \)

\( f_2 = f(x_2, y_2) = \frac{2 - 1.0143^2}{5(4.2)} = 0.0452 \)

\( \frac{54}{12} \)
By Adam's - Bashforth predictor formula \( y_f^{(0)} = y_0 + \frac{h}{2} [f(x_0, y_0) + 3f(x_1, y_1)] \)

\[ = 1.0183 + \frac{0.1}{2} [f(0,0.043) - 5f(0.043) + 3f(0.046) - 9(0.046)] \]

\[ = 1.0187. \]

Now \( y_f = f(x, y_f^{(0)}) = f(x, 1.0187) = 0.0437 \)

By Adam's - Moulton corrector formula \( y_0^{(0)} = y_0 + \frac{h}{2} [f(x_0, y_0) + 3f(x_1, y_1)] \)

\[ y_0^{(0)} = 1.0187. \]

Since \( y_0^{(0)} = y_0^{(0)}, \) \( y_f = 1.0187 \)

Hence \( y(0.04) = 1.0187. \)

3. If \( \frac{dy}{dx} = 2e^x y; \) \( y(0) = 2, \) find \( y(0.4) \) using Adam's Predictor - corrector formula by calculating \( y(0.1), y(0.2), y(0.3) \) using Euler - modified method.

Here \( f(x, y) = 2e^x y, \) \( h = 0.1, x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.2 \)

To find \( y_1 = y_0 + h \cdot f(x_0, y_0) \)

By Modified Euler's method \( y_1 = y_0 + h \cdot f(x_0, y_0) \)

By Euler's formula, \( y_1^{(0)} = y_0 + h \cdot f(x_0, y_0) \)

For \( n = 1: \)

\( y_1 = 2 + 0.1 \cdot \left[ 4 + f(x_1, y_1^{(0)}) \right] = 2 + 0.1 \cdot \left[ 4 + 2e^{0.1} \times 2.4 \right] = 2.4662 \)

\( y_1^{(0)} = 2 + 0.1 \cdot \left[ 4 + f(x_1, y_1^{(0)}) \right] = 2 + 0.1 \cdot \left[ 4 + 2e^{0.1} \times 2.4662 \right] = 2.4662 \)

\( y_1 = 2 + 0.1 \cdot \left[ 4 + f(x_1, y_1^{(0)}) \right] = 2 + 0.1 \cdot \left[ 4 + 2e^{0.1} \times 2.4662 \right] = 2.4662 \)

\( y_1^{(0)} = 2 + 0.1 \cdot \left[ 4 + f(x_1, y_1^{(0)}) \right] = 2 + 0.1 \cdot \left[ 4 + 2e^{0.1} \times 2.4662 \right] = 2.4662 \)

\( y_1 = 2 + 0.1 \cdot \left[ 4 + f(x_1, y_1^{(0)}) \right] = 2 + 0.1 \cdot \left[ 4 + 2e^{0.1} \times 2.4662 \right] = 2.4662 \)

\( y_1 \approx y_1^{(0)}; \) \( y_1 = y(0.1) = 2.4662 \)

To find \( y_2 \) i.e. \( y(0.2): \)

By Euler's formula, \( y_2^{(0)} = y_1 + h \cdot f(x_1, y_1) \)

\( f(x_1, y_1) = 2e^{0.1} \times 2.4662 \)

\( = 2.9333 + 0.1 \times 5.9333 \)

\( = 2.0200 \)
\[ y_2^{(0)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_1, y_0^{(0)}) \right] \]

\[ y_2^{(1)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_1, y_1^{(0)}) \right] \]

\[ y_3^{(1)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_1, y_1^{(0)}) \right] \]

\[ y_3^{(2)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_1, y_1^{(0)}) \right] \]

\[ y_3^{(3)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_1, y_1^{(0)}) \right] \]

\[ y_3^{(4)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_1, y_1^{(0)}) \right] \]

\[ y_3^{(5)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_1, y_1^{(0)}) \right] \]

\[ y_3^{(6)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_1, y_1^{(0)}) \right] \]

\[ y_3^{(7)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_1, y_1^{(0)}) \right] \]

\[ y_3^{(8)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_1, y_1^{(0)}) \right] \]

\[ y_3^{(9)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_1, y_1^{(0)}) \right] \]

\[ y_3^{(10)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_1, y_1^{(0)}) \right] \]

\[ y_3^{(11)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_1, y_1^{(0)}) \right] \]

\[ y_3^{(12)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_1, y_1^{(0)}) \right] \]

\[ y_3^{(13)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_1, y_1^{(0)}) \right] \]

\[ y_3^{(14)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_1, y_1^{(0)}) \right] \]

\[ y_3^{(15)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_1, y_1^{(0)}) \right] \]

\[ y_3^{(16)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_1, y_1^{(0)}) \right] \]

\[ y_3^{(17)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_1, y_1^{(0)}) \right] \]

\[ y_3^{(18)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_1, y_1^{(0)}) \right] \]

\[ y_3^{(19)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_1, y_1^{(0)}) \right] \]

\[ y_3^{(20)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_1, y_1^{(0)}) \right] \]

To find \( y_0 \): \( y_0 = y(0.3) = 3.1284 \)

To find \( y_1 \): \( y_1 = y(0.3) = 3.1284 \)

To find \( y_2 \): \( y_2 = y(0.3) = 3.1284 \)

To find \( y_3 \): \( y_3 = y(0.3) = 3.1284 \)

To find \( y_4 \): \( y_4 = y(0.3) = 3.1284 \)

To find \( y_5 \): \( y_5 = y(0.3) = 3.1284 \)

To find \( y_6 \): \( y_6 = y(0.3) = 3.1284 \)

To find \( y_7 \): \( y_7 = y(0.3) = 3.1284 \)

To find \( y_8 \): \( y_8 = y(0.3) = 3.1284 \)

To find \( y_9 \): \( y_9 = y(0.3) = 3.1284 \)

To find \( y_{10} \): \( y_{10} = y(0.3) = 3.1284 \)

To find \( y_{11} \): \( y_{11} = y(0.3) = 3.1284 \)

To find \( y_{12} \): \( y_{12} = y(0.3) = 3.1284 \)

To find \( y_{13} \): \( y_{13} = y(0.3) = 3.1284 \)

To find \( y_{14} \): \( y_{14} = y(0.3) = 3.1284 \)

To find \( y_{15} \): \( y_{15} = y(0.3) = 3.1284 \)

To find \( y_{16} \): \( y_{16} = y(0.3) = 3.1284 \)

To find \( y_{17} \): \( y_{17} = y(0.3) = 3.1284 \)

To find \( y_{18} \): \( y_{18} = y(0.3) = 3.1284 \)

To find \( y_{19} \): \( y_{19} = y(0.3) = 3.1284 \)

To find \( y_{20} \): \( y_{20} = y(0.3) = 3.1284 \)
By Adam's predictor formula, $y_4 = y_{1 + \frac{3}{2}f}\left[55f_3 - 59f_1 + 39f_1 - 9f_0\right]$

Now $f_4 = f(x_1, y_1^{(4)}) = f(0.4, 15.3888) = 16.0634$

By Adam's Muller corrector method, $y_4^{(1)} = y_2 + \frac{b_2}{2\beta}[9f_4 + 19f_3 - 5f_2 + f_1]$

Now $f_4 = f(x_1, y_4^{(1)}) = f(0.4, 5.9392) = 16.0884$

By Adam's Muller corrector method, $y_4^{(2)} = y_3 + \frac{b_3}{2\beta}[9f_4 + 19f_3 - 5f_2 + f_1]$

Now $f_4 = f(x_1, y_4^{(2)}) = f(0.4, 5.9392) = 16.0914$

Now $f_4 = f(x_4, y_4^{(3)}) = f(0.4, 5.9393) = 16.0914$

Using Adam's Muller corrector method, $y_4^{(2)} = y_3 + \frac{b_3}{2\beta}[9f_4 + 19f_3 - 5f_2 + f_1]$

Now $f_4 = f(x_4, y_4^{(4)}) = f(0.4, 5.9393) = 16.0914$

Using Adam's Muller corrector method, $y_4 = 5.9393$

$\therefore y_4 = y_4^{(4)}$; $y_4 = y(0.4) = 5.9393$

\(Q\) If $y_0 = 0$, $y_1 = 10$, $y_2 = \ldots$