Formation of partial differential equations by eliminating arbitrary constants

Partial Differential Equation: An equation involving a dependent variable and its derivatives with respect to two or more independent variables is known as a partial differential equation (PDE).

Order of a PDE: The order of a PDE is defined as the order of the highest partial derivative occurring in the PDE.

Degree of a PDE: The degree of a PDE is the degree of the highest partial derivative which occurs in it after the equation has been rationalized, i.e., made free from radicals and fractions so far as derivatives are concerned.

Examples: i) \( x \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} = 3u \) is the PDE of the first order and first degree

ii) \( \frac{\partial^2 z}{\partial x^2} = \left( 1 + \frac{\partial z}{\partial y} \right)^{1/2} \) is the PDE of second order and second degree

Notations: Let \( z = f(x, y) \) be the function of two independent variables \( x \) and \( y \), then we adopt the following notations throughout the study of partial differential equations: \( p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y} \) and \( t = \frac{\partial^2 z}{\partial y^2} \)

Formation of PDE: Partial differential equations can be formed either by the elimination of arbitrary constants or arbitrary functions from a relation involving 3 or more variables.

Formation of PDE by eliminating arbitrary constants:

Consider \( F(x, y, z, a, b) = 0 \) ... (1), where \( a, b \) are arbitrary constants. Let \( z \) be regarded as function of two independent variables \( x \) and \( y \). Differentiating (1) with respect \( x \) and \( y \) partially in turn, we get

\[
\frac{\partial F}{\partial x} + p \frac{\partial F}{\partial z} = 0 \quad \text{... (2)}
\]

\[
\frac{\partial F}{\partial y} + q \frac{\partial F}{\partial z} = 0 \quad \text{... (3)}
\]

Eliminating two arbitrary constants \( a \) and \( b \) from (1), (2) and (3), we shall obtain an equation of the form \( f(x, y, z, p, q) = 0 \) ... (4), which is PDE of the first order.
Note: i) When the number of arbitrary constants is less than the number of independent variables, then the elimination of arbitrary constants usually gives rise to more than one PDE of order one.

ii) When the number of arbitrary constants is equal to the number of independent variables, then the elimination of arbitrary constants shall give rise to a unique PDE of first order.

ii) When the number of arbitrary constants is greater than the number of independent variables, then the elimination of arbitrary constants leads to a PDE of order usually greater than one.

Example 1.1. Derive a PDE by eliminating the constants $a$ and $b$ from $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Solution. Given $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ … (1)

Differentiating (1) partially with respect to $x$,

$$2 \frac{\partial z}{\partial x} = \frac{1}{a^2}(2x) \Rightarrow \frac{1}{a^2} = \frac{p}{x} \quad \ldots (2)$$

Differentiating (1) partially with respect to $y$,

$$2 \frac{\partial z}{\partial y} = \frac{1}{b^2}(2y) \Rightarrow \frac{1}{b^2} = \frac{q}{y} \quad \ldots (3)$$

Substituting the values of $a$ and $b$ from (2) and (3) in (1), we get

$$2z = x^2 \left(\frac{p}{x}\right) + y^2 \left(\frac{q}{y}\right)$$

or

$$px + qy = 2z$$,

which is the required partial differential equation.

Example 1.2. Find the PDE of all planes which are at a constant distance $a$ from the origin.

Solution. The equation of the plane in normal form is $lx + my + nz = a$ … (1)

where $l, m, n$ are direction cosines of the normal so that $l^2 + m^2 + n^2 = 1$ … (2)

Differentiating (1) partially with respect to $x$ and $y$, we get

$$l + n \frac{\partial z}{\partial x} = 0 \Rightarrow l = -np \quad \text{and} \quad m + n \frac{\partial z}{\partial y} = 0 \Rightarrow m = -nq \quad \ldots (3)$$

Substituting the values of $l, m$ from (3) in (2), we get

$$n^2 (p^2 + q^2 + 1) = 1 \Rightarrow n = \frac{1}{\sqrt{p^2 + q^2 + 1}} \quad \ldots (4)$$

From (3) and (4), $l = -\frac{p}{\sqrt{p^2 + q^2 + 1}}$ \quad \text{and} \quad $m = -\frac{q}{\sqrt{p^2 + q^2 + 1}} \ldots (5)$
Substituting the values of \( l, m, n \) given by (4) and (5) in (1), we get

\[
\frac{p_x}{\sqrt{p^2+q^2+1}} - \frac{q_y}{\sqrt{p^2+q^2+1}} + \frac{z}{\sqrt{p^2+q^2+1}} = a
\]

\[
\Rightarrow p_x + q_y - z = a\sqrt{p^2 + q^2 + 1}
\]

\[
\Rightarrow z = p_x + q_y + a\sqrt{p^2 + q^2 + 1}, \text{which is the required partial differential equation.}
\]

**Exercise 1**

Form the PDE by eliminating the arbitrary constants from:

1. \( z = a \log \left( \frac{b(y-1)}{x-1} \right) \)
2. \( \log(a z - 1) = x + ay + b \)
3. \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \)
4. \( z = ax + by + a^2 + b^2 \)
5. \( z = ax^3 + by^3 \)
6. \( z = Ae^{pt} \sin px \)
7. \( z = (x^2 + a)(y^2 + b) \)
8. \( z = axe^y + \frac{1}{2} a^2 e^{2y} + b \)
9. \( z = xy + y\sqrt{x^2 + a^2 + b} \)
10. \( (x-a)^2 + (y-b)^2 = z^2 \cot^2 \alpha \) (\( \alpha \) is fixed constant)
11. Find the PDE of all spheres of fixed radius having their centers in the \( xy \)-plane.
12. Find the differential equation of all spheres whose centers lie on the \( z \)-axis.

**Answers:**

1. \( px + qy = p + q \)
2. \( p(l + q) = zf \)
3. \( xp^2 - pz + xzr = 0 \)
4. \( z = xp + yq + p^2 + q^2 \)
5. \( px + qy = 3z \)
6. \( \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \)
7. \( pq = 4xyz \)
8. \( px + p^2 - q = 0 \)
9. \( xp + yq = pq \)
10. \( p^2 + q^2 = \tan^2 \alpha \)
11. \( (p^2 + q^2 + 1)z^2 = r^2 \)
12. \( yp - xq = 0 \)
A4011 - PARTIAL DIFFERENTIAL EQUATIONS AND COMPLEX VARIABLES
Handout # 2

Formation of partial differential equation by eliminating arbitrary functions
1. If the given relation between the variables \( x, y, z \) contains only a single arbitrary function, then the elimination of arbitrary function gives rise to a PDE of first order.
2. If the given relation between the variables \( x, y, z \) contains two arbitrary functions, then the elimination of arbitrary functions leads to PDE of higher order.

Example 2.1. Form the partial differential equations by eliminating arbitrary function \( \phi \) from \( z = (x + y)\phi(x^2 - y^2) \)

Solution. Given \( z = (x + y)\phi(x^2 - y^2) \) . . . (1)

Differentiating (1) partially with respect to \( x \),

\[ \frac{\partial z}{\partial x} = (x + y)\phi'(x^2 - y^2)(2x) + \phi(x^2 - y^2) \]
\[ \Rightarrow p - \phi(x^2 - y^2) = (x + y)\phi'(x^2 - y^2)(2x) \]
\[ \Rightarrow p - \frac{x^2 - y^2}{x + y} = 2x(x + y)\phi'(x^2 - y^2) \] . . . (2)

Differentiating (1) partially with respect to \( y \),

\[ \frac{\partial z}{\partial y} = (x + y)\phi'(x^2 - y^2)(-2y) + \phi(x^2 - y^2) \]
\[ \Rightarrow q - \phi(x^2 - y^2) = (x + y)\phi'(x^2 - y^2)(-2y) \]
\[ \Rightarrow q - \frac{x^2 - y^2}{x + y} = -2y(x + y)\phi'(x^2 - y^2) \] . . . (3)

(2) \( \div \) (3) gives
\[ \frac{p - \frac{x^2 - y^2}{x + y}}{q - \frac{x^2 - y^2}{x + y}} = -\frac{2x}{2y} \]
\[ \Rightarrow yp - \frac{y(x)}{x + y} = -xq + \frac{xz}{x + y} \]
\[ \Rightarrow yp + xq = z \]

Example 2.2. Form the partial differential equation by eliminating the arbitrary functions \( f, g \) from \( z = f(x + at) + g(x - at) \)

Solution. Given \( z = f(x + at) + g(x - at) \) . . . (1)

Differentiating (1) partially with respect to \( x \),
\[ \frac{\partial z}{\partial t} = f'(x+at) + g'(x-at) \ldots (2) \]

Differentiating (1) partially with respect to \( t \),
\[ \frac{\partial z}{\partial t} = af'(x+at) - ag'(x-at) \ldots (3) \]

Differentiating (2) partially with respect to \( x \),
\[ \frac{\partial^2 z}{\partial x^2} = f''(x+at) + g''(x-at) \ldots (4) \]

Differentiating (3) partially with respect to \( t \),
\[ \frac{\partial^2 z}{\partial t^2} = a^2 f''(x+at) + a^2 g''(x-at) \]
\[ \Rightarrow \frac{\partial^2 z}{\partial t^2} = a^2 \left[ f''(x+at) + g''(x-at) \right] \]
\[ \Rightarrow \frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2} \quad \text{[\because By (4)]} \]

**Exercise 2**

Form the partial differential equations by eliminating the arbitrary function(s) from:

1. \( z = y^2 + 2f\left(\frac{1}{x} + \log y\right) \)
2. \( xyz = \phi(x+y+z) \)
3. \( F(xy+z^2, x+y+z) = 0 \)
4. \( z = yf(x) + xg(y) \)
5. \( z = f(x) + e^z g(x) \)
6. \( f(x^2 + y^2, z - xy) = 0 \)
7. \( F(x+y+z, x^2+y^2+z^2) = 0 \)
8. \( z = e^{xy} \phi(x-y) \)
9. \( z = f_1(y+2x) + f_2(y-3x) \)
10. \( z = f_1(x)f_2(y) \)
11. \( z = f\left(\frac{xy}{z}\right) \)
12. \( z = xf_1(x+t) + f_2(x+t) \)

**Answers:**

1. \( x^2 p + yq = 2y^2 \)
2. \( x(y-z) p + y(z-x) q = z(x-y) \)
3. \( (2z-x) p + (y-2z) q = x-y \)
4. \( xys - xp - yq + z = 0 \)
5. \( q-t = 0 \)
6. \( yp - xq = y^2 - x^2 \)
7. \( (y-z) p + (z-x) q = x-y \)
8. \( p + q = mz \)
9. \( r + s - 6t = 0 \)
10. \( zs - pq = 0 \)
11. \( xp - yq = 0 \)
12. \[ \frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial t} + \frac{\partial^2 z}{\partial t^2} = 0 \]
Solutions of a partial differential equation

**Complete solution:** A solution of a PDE in which the number of arbitrary constants is equal to the number of independent variables is called its complete solution.

**e.g.** \( z = ax^3 + by^3 \) is the complete solution of \( px + qy = 3z \)

**Particular solution:** A solution of a PDE which is obtained from the complete solution by assigning particular values to the arbitrary constants is called a particular solution.

**e.g.** \( z = x^3 + 2y^3 \) is a particular solution of \( px + qy = 3z \)

**General solution:** A solution of a PDE in which the number of arbitrary functions is equal to the order of the PDE is called its general solution.

**e.g.** \( z = f_1(y + 2x) + f_2(y - 3x) \) is the general solution of \( r + s - 6t = 0 \)

**Classification of first order PDE**

**Linear PDE:** A first order PDE \( f(x, y, z, p, q) = 0 \) is said to be linear if the dependent variable \( z \) and its partial derivatives \( p, q \) occur only in the first degree and products of \( z, p \) and \( q \) do not appear in the equation.

**Example:** \( x^2 yp + xy^2 q = xyz \)

**Quasi-linear PDE:** A first order PDE \( f(x, y, z, p, q) = 0 \) is said to be quasi-linear if the partial derivatives \( p, q \) occur only in the first degree and products of \( p, q \) do not appear in the equation.

**Example:** \( xy p + yz q = z x \)

**Non-linear PDE:** A first order PDE \( f(x, y, z, p, q) = 0 \) is said to be non-linear if it contains \( p, q \) of degree other than one and/or product terms of \( p, q \) appear in the equation.

**Example:** i) \( \sqrt{p} + \sqrt{q} = 1 \) ii) \( p^2 z^2 + q^2 = p^2 q \)

**Lagrange’s equation:** A quasi-linear PDE of first order of the form \( Pp + Qq = R \), where \( P, Q \) and \( R \) are functions of \( x, y, z \), is known as Lagrange’s equation.

**Example:** \( (z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx \)
Working rule for solving Lagrange’s equation:

i) Write the given linear PDE of first order in the standard form \( Pp + Qq = R \)

\[ (1) \]

ii) Write Lagrange’s auxiliary equations

\[ \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \]  

\[ (2) \]

iii) Solve (2) by using either the method of grouping or the method of multipliers. Let

\[ u(x, y, z) = c_1 \] and \( v(x, y, z) = c_2 \) be two linearly independent solutions of (2).

iv) The general solution of (1) is then written in one of the following three forms:

\[ F(u, v) = 0 \quad \text{or} \quad u = \phi(v) \quad \text{or} \quad v = \psi(u) \]

**Method of grouping:** Suppose that one of the variables is either absent or cancels out from any two fractions of (2). Then an integral (solution) can be obtained by variables separable method. The same procedure can be repeated with another set of two fractions of (2).

**Method of multipliers:** Introducing Lagrange’s multipliers \( l_1, m_1, n_1 \) which are functions of \( x, y, z \) or constants, each fraction in (2) is equal to

\[ \frac{l_1 dx + m_1 dy + n_1 dz}{l_1 P + m_1 Q + n_1 R} \]  

\[ (3) \]

If \( l_1, m_1, n_1 \) are so chosen that \( l_1 P + m_1 Q + n_1 R = 0 \) then we get \( l_1 dx + m_1 dy + n_1 dz = 0 \)

Integrating this, we get \( u(x, y, z) = c_1 \)

Similarly choose another set of multipliers \( l_2, m_2, n_2 \) in such a way that

\[ l_2 P + m_2 Q + n_2 R = 0 \] then we get \( l_2 dx + m_2 dy + n_2 dz = 0 \)

Integrating this, we get \( v(x, y, z) = c_2 \)

Write the general solution of (1) as \( F(u, v) = 0 \)

**Note:** Multipliers may be chosen such that the numerator \( l_1 dx + m_1 dy + n_1 dz \) is an exact differential of the denominator \( l_1 P + m_1 Q + n_1 R \). Now combine (3) with a fraction of (2) to get an integral (solution).

**Example 3.1.** Solve \( \frac{y^2 z}{x} p + xzq = y^2 \)

**Solution.** Given \( \frac{y^2 z}{x} p + xzq = y^2 \)
The Lagrange’s auxiliary equations are \[ \frac{dx}{y^2z} = \frac{dy}{xz} = \frac{dz}{y^2} \]

Taking the first two fractions, we get
\[ \frac{dx}{y^2z} = \frac{dy}{xz} \]
\[ \Rightarrow x^2dx = y^2dy \]

Integrating, \( x^3 - y^3 = c_1 \)

Again taking the first and the last fractions, we get
\[ \frac{dx}{y^2z} = \frac{dz}{y^2} \]
\[ \Rightarrow xdx = zdz \]

Integrating, \( x^2 - z^2 = c_2 \)

The general solution is \( F(x^3 - y^3, x^2 - z^2) = 0 \)

**Example 3.2.** Solve \( p - q = \log(x + y) \)

**Solution.** The Lagrange’s auxiliary equations are \[ \frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\log(x + y)} \]

Taking the first two fractions, we get
\[ dx = -dy \]

Integrating, \( x + y = c_1 \)

Again taking the first and the last fractions, we get \( dx = \frac{dz}{\log(x + y)} \)
\[ \Rightarrow \log(x + y)dx = dz \]
\[ \Rightarrow \log c_1dx = dz \]
\[ (\because x + y = c_1) \]

Integrating, \( x\log c_1 = z + c_2 \) or \( x\log(x + y) - z = c_2 \)

Therefore, the general solution is \( F(x + y, x\log(x + y) - z) = 0 \)

**Example 3.3.** Solve \( y^2p - xyq = x(z - 2y) \)

**Solution.** The Lagrange’s auxiliary equations are \[ \frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z - 2y)} \]

Taking the first two fractions, we get
\[ \frac{dx}{y^2} = \frac{dy}{-xy} \]
\[ \Rightarrow xdx = - ydy \]

Integrating, \( x^2 + y^2 = c_1 \)

Choosing 2x, z, y as multipliers then

\[ 2xdx + zdy + ydz = 2xy^2 - xyz + xy(z - 2y) \]

\[ \Rightarrow 2xdx + zdy + ydz = 2xy^2 - xyz + xy - 2xy^2 \]

\[ \Rightarrow 2xdx + d(yz) = 0 \]

Integrating, \( x^2 + yz = c_2 \)

The general solution is \( F(x^2 + y^2, x^2 + yz) = 0 \)

**Example 3.4.** Solve \( x(y - z)p + y(z - x)q = z(x - y) \)

**Solution.** The Lagrange’s auxiliary equations are

\[ \frac{dx}{x(y - z)} = \frac{dy}{y(z - x)} = \frac{dz}{z(x - y)} \]

Choosing 1, 1, 1 as multipliers then

\[ dx + dy + dz = x(y - z) + y(z - x) + z(x - y) \]

\[ \Rightarrow dx + dy + dz = xy - xz + yz - xy + zx - zy = 0 \]

Integrating, \( x + y + z = c_1 \)

Choosing \( \frac{1}{x}, \frac{1}{y}, \frac{1}{y} \) as another set of multipliers then

\[ \frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz = \frac{1}{x} \cdot x(y - z) + \frac{1}{y} \cdot y(z - x) + \frac{1}{z} \cdot z(x - y) \]

\[ \Rightarrow \frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz = y - z + z - x + x - y = 0 \]

Integrating, \( \log x + \log y + \log z = \log c_2 \) or \( xyz = c_2 \)

The general solution is \( F(x + y + z, xyz) = 0 \)

**Example 3.5.** Solve \( (x^2 - yz)p + (y^2 - zx)q = z^2 - xy \)

**Solution:** The Lagrange’s auxiliary equations are

\[ \frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy} \] (1)

Choosing \( y + z, z + x, x + y \) as multipliers then

\[ (y + z)dx + (z + x)dy + (x + y)dz = (y + z)(x^2 - yz) + (z + x)(y^2 - zx) + (x + y)(z^2 - xy) \]

\[ \Rightarrow (ydx + xdy) + (zdy + ydz) + (xdz + zdx) = x^2 y - y^2 z + x^2 z - yz^2 + y^2 z - xz^2 + xy^2 - x^2 z \]

\[ + xz^2 - x^2 y + yz^2 - xy^2 \]

\[ \Rightarrow d(xy) + d(yz) + d(zx) = 0 \]

Integrating, \( xy + yz + zx = c_1 \)

Choosing 1, -1, 0 and 0, 1, -1 as two sets of multipliers, then
Each fraction of (1) = \[ \frac{1 \cdot dx - 1 \cdot dy + 0 \cdot dz}{(x^2 - yz) - (y^2 - zx) + 0} = \frac{0 \cdot dx + 1 \cdot dy - 1 \cdot dz}{0 + (y^2 - zx) - (z^2 - xy)} \]

\[ \Rightarrow \frac{dx - dy}{x^2 - y^2 + z(x - y)} = \frac{dy - dz}{y^2 - z^2 + x(y - z)} \]

\[ \Rightarrow \frac{dx - dy}{(x - y)(x + y + z)} = \frac{dy - dz}{(y - z)(x + y + z)} \]

\[ \Rightarrow \frac{d(x - y)}{x - y} = \frac{d(y - z)}{y - z} \]

Integrating, log(x - y) = log(y - z) + log c \quad \text{or} \quad \frac{x - y}{y - z} = c_2

The general solution is \( F\left( xy + yz + zx, \frac{x - y}{y - z}\right) = 0 \)

**Exercise 3**

Solve the following partial differential equations:

1. \( xp + yq = 3z \)
2. \( p\sqrt{x} + q\sqrt{y} = \sqrt{z} \)
3. \( p \tan x + q \tan y = \tan z \)
4. \( yzp + xzq = xy \)
5. \( z(z^2 + xy)(px - qy) = x^4 \)
6. \( x^2 p + y^2 q = z(x + y) \)
7. \( xp - yq = y^2 - x^2 \)
8. \( (z - y)p + (x - z)q = y - x \)
9. \( (y^2 + z^2)p - xyq + zx = 0 \)
10. \( x^2(y - z)p + y^2(z - x)q = z^2(x - y) \)
11. \((mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} = ly - mx \)
12. \( x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2) \)
13. \( (y + zx)p - (x + yz)q = x^2 - y^2 \)
14. \( (z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx \)
15. \( (x^2 - y^2 - z^2)p + 2xyz = 2xz \)

**Answers:**

1. \( F\left( \frac{x}{y}, \frac{x^3}{z}\right) = 0 \)
2. \( F\left( \sqrt{x} - \sqrt{y}, \sqrt{y} - \sqrt{z}\right) = 0 \)
3. \( F\left( \frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0 \)
4. \( F\left( x^2 - y^2, y^2 - z^2\right) = 0 \)
5. \( F\left( xy, x^4 - 2xyz^2 - z^4\right) = 0 \)
6. \( F\left( \frac{1}{x} - \frac{1}{y}, \frac{xy}{z}\right) = 0 \)
7. \( F\left( xy, x^2 + y^2 + 2z\right) = 0 \)
8. \( F\left( x + y + z, x^2 + y^2 + z^2\right) = 0 \)
9. \( F\left( \frac{y}{z}, x^2 + y^2 + z^2\right) = 0 \)
10. \( F\left( xyz, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0 \)
11. \( F\left( lx + my + nz, x^2 + y^2 + z^2\right) = 0 \)
12. \( F\left( xyz, x^2 + y^2 - 2z\right) = 0 \)
13. \( F\left( xy + z, x^2 + y^2 - z^2\right) = 0 \)
14. \( F\left( y^2 - 2yz - z^2, x^2 + y^2 + z^2\right) = 0 \)
15. \( F\left( \frac{y}{z}, x^2 + y^2 + z^2\right) = 0 \)
Non-linear partial differential equations of first order

Non-linear partial differential equations of first order contains \( p \) and \( q \) of degree other than one and/or product terms of \( p \) and \( q \). Its complete solution is given by \( \phi(x,y,z,a,b) = 0 \). In this section, we shall discuss four standard forms of these equations.

**Form I.** \( f(p,q) = 0 \), i.e., equations containing \( p \) and \( q \) only.

Let the required solution be \( z = ax + by + c \) . . . (1)

Then \( p = \frac{\partial z}{\partial x} = a \) and \( q = \frac{\partial z}{\partial y} = b \)

Substituting these values of \( p \) and \( q \) in \( f(p,q) = 0 \), we get

\[ f(a,b) = 0 \] . . . (2)

Expressing (2) as \( b = \phi(a) \) and substituting this value of \( b \) in (1), we get the required solution as

\[ z = ax + \phi(a)y + c \], where \( a \), \( c \) are arbitrary constants.

**Example 4.1.** Solve \( p + q = pq \)

**Solution.** Given \( p + q = pq \) . . . (1)

Let the required solution be \( z = ax + by + c \) . . . (2)

Then \( p = \frac{\partial z}{\partial x} = a \) and \( q = \frac{\partial z}{\partial y} = b \)

Substituting these values of \( p \) and \( q \) in (1), we get

\[ a + b = ab \Rightarrow b = \frac{a}{a-1} \]

Substituting the value of \( b \) in (2), we get

\[ z = ax + \frac{a}{a-1} y + c \], which is the required solution.

**Example 4.2.** Solve \( p^2 + q^2 = npq \)

**Solution.** Given \( p^2 + q^2 = npq \) . . . (1)

Let the required solution be \( z = ax + by + c \) . . . (2)

Then \( p = \frac{\partial z}{\partial x} = a \) and \( q = \frac{\partial z}{\partial y} = b \)

Substituting these values of \( p \) and \( q \) in (1), we get

\[ a^2 + b^2 = nab \Rightarrow b^2 - nab + a^2 = 0 \], which is quadratic in \( b \)
\[ b = \frac{na \pm \sqrt{n^2 a^2 - 4a^2}}{2} = \frac{a}{2} \left( n \pm \sqrt{n^2 - 4} \right) \]

Substituting the value of \( b \) in (2), we get \( z = ax + \frac{a}{2} \left( n \pm \sqrt{n^2 - 4} \right) y + c \)

**Form II.** \( f(z, p, q) = 0 \), i.e., equations not containing \( x \) and \( y \)

Assume \( q = ap \) . . . (1)

Substituting \( q = ap \) in given PDE, we get \( f(z, p, ap) = 0 \) . . . (2)

Solving (2) for \( p \), we get \( p = \phi(z) \) . . . (3)

We know that \( dz = pdx + qdy = p(dx + ady) \) \( \therefore q = ap \)

\[ \Rightarrow dz = \phi(z)(dx + ady) \quad \text{[} \therefore \text{ By (3)} \text{]} \]

\[ \Rightarrow \frac{dz}{\phi(z)} = dx + ady \]

Integrating on both sides, we get

\[ \int \frac{dz}{\phi(z)} = x + ay + c, \text{ where } a, c \text{ are arbitrary constants.} \]

**Example 4.3.** Solve \( zpq = p + q \)

**Solution.** Given \( zpq = p + q \) . . . (1)

Assume \( q = ap \)

Substituting \( q = ap \) in (1), we get

\[ zp(ap) = p + ap \Rightarrow p = \frac{1 + a}{az} \quad \text{. . . (2)} \]

We know that \( dz = pdx + qdy = p(dx + ady) \) \( \therefore q = ap \)

\[ \Rightarrow dz = \frac{1 + a}{az}(dx + ady) \quad \text{[} \therefore \text{ By (2)} \text{]} \]

\[ \Rightarrow \frac{dz}{az} = \frac{1 + a}{az} dx + ady \]

Integrating, \( \frac{az^2}{2(1 + a)} = x + ay + c, \text{ where } a, c \text{ are arbitrary constants.} \)

**Example 4.4.** Solve \( z = p^2 + q^2 \)

**Solution.** Given \( z = p^2 + q^2 \) . . . (1)

Assume \( q = ap \)

Substituting \( q = ap \) in (1), we get

\[ z = p^2 + a^2p^2 \Rightarrow p = \sqrt{\frac{z}{1 + a^2}} \quad \text{. . . (2)} \]

We know that \( dz = pdx + qdy = p(dx + ady) \) \( \therefore q = ap \)

\[ \Rightarrow dz = \sqrt{\frac{z}{1 + a^2}}(dx + ady) \quad \text{[} \therefore \text{ By (2)} \text{]} \]
\[ \Rightarrow \sqrt{1+a^2} \, dz = dx + ady \]

Integrating, \( 2\sqrt{z(1+a^2)} = x + ay + c \), where \( a, c \) are arbitrary constants.

**Form III.** \( f(x, p) = g(y, q) \), i.e., equations in which \( z \) is absent and the terms containing \( x \) and \( p \) can be separated from those containing \( y \) and \( q \).

Assume \( f(x, p) = g(y, q) = a = \text{constant} \)

\[ \therefore f(x, p) = a \quad \text{and} \quad g(y, q) = a \ldots (1) \]

Solving each equation in (1) for \( p \) and \( q \), we get

\[ p = f_1(x, a) \quad \text{and} \quad q = g_1(y, a) \]

We know that \( dz = p \, dx + q \, dy \)

\[ \Rightarrow dz = f_1(x, a) \, dx + g_1(y, a) \, dy \]

Integrating on both sides, we get

\[ z = \int f_1(x, a) \, dx + \int g_1(y, a) \, dy + c \], where \( a, c \) are arbitrary constants.

**Example 4.5.** Solve \( yp + xq + pq = 0 \)

**Solution.** Rewriting \((x + p)q = -yp\)

or \( \frac{x+p}{p} = -\frac{y}{q} = a \), say

\[ \therefore \frac{x+p}{p} = a \quad \text{and} \quad \frac{y}{q} = -a \]

Solving for \( p \) and \( q \), we get \( p = \frac{x}{a-1} \quad \text{and} \quad q = -\frac{y}{a} \)

We know that \( dz = p \, dx + q \, dy \)

\[ \Rightarrow dz = \frac{x}{a-1} \, dx - \frac{y}{a} \, dy \]

Integrating, \( z = \frac{x^2}{2(a-1)} - \frac{x^2}{2a} + c \), where \( a, c \) are arbitrary constants.

**Example 4.6.** Solve \( p^2 + q^2 = x^2 + y^2 \)

**Solution.** Rewriting \( p^2 - x^2 = y^2 - q^2 = a \), say

\[ \therefore p^2 - x^2 = a \quad \text{and} \quad y^2 - q^2 = a \]

Solving for \( p \) and \( q \), we get \( p = \sqrt{x^2 + a} \quad \text{and} \quad q = \sqrt{y^2 - a} \)

We know that \( dz = p \, dx + q \, dy \Rightarrow dz = \sqrt{x^2 + a} \, dx + \sqrt{y^2 - a} \, dy \)

Integrating, we get

\[ z = \frac{x}{2} \sqrt{x^2 + a} + \frac{a}{2} \log(x + \sqrt{x^2 + a}) + \frac{y}{2} \sqrt{y^2 - a} - \frac{a}{2} \log(y + \sqrt{y^2 - a}) + c \]
or \[ z = \frac{x^2}{2} + a + \frac{y^2}{2} - a + \frac{2}{a} \log \left( \frac{x + \sqrt{x^2 + a}}{y + \sqrt{y^2 - a}} \right) + c, \] where \( a, c \) are arbitrary constants

**Form IV.** \( z = px + qy + f(p, q) \) [Clairaut’s equation]

The complete solution of Clairaut’s equation is \( z = ax + by + f(a, b) \) which is obtained by writing \( a \) for \( p \) and \( b \) for \( q \) in the given equation.

**Example 4.7.** Solve \( z = px + qy + \log pq \)

**Solution.** Given PDE is a Clairaut’s equation and its complete solution is given by

\[ z = ax + by + \log ab, \] where \( a, b \) are arbitrary constants

**Example 4.8.** Solve \((p-q)(z-px-qy) = 1\)

**Solution.** Given PDE can be written as \( z - px - qy = \frac{1}{p-q} \)

or \[ z = px + qy + \frac{1}{p-q}, \] which is Clairaut’s equation

\[ \therefore \] Its complete solution is \( z = ax + by + \frac{1}{a-b}, \) where \( a, b \) are arbitrary constants

**Exercise 4**

Solve the following:

1. \( \sqrt{p} + \sqrt{q} = 1 \)
2. \( p^2 + q^2 = m^2 \)
3. \( pq = k \)
4. \( p(1 + q) = qz \)
5. \( p^2z^2 + q^2 = p^2q \)
6. \( q^2 = z^2p^2(1 - p^2) \)
7. \( p(1 - q^2) = q(1 - z) \)
8. \( p + q = \sin x + \sin y \)
9. \( \sqrt{p} + \sqrt{q} = x + y \)
10. \( p^2 - q^2 = x - y \)
11. \( pqz = p^2(qx + p^2) + q^2(py + q^2) \)
12. \( z = px + qy - 2\sqrt{pq} \)

**Answers:**

1. \( z = ax + \left(1 - \sqrt{a}\right)^2 y + c \)
2. \( z = ax + \sqrt{m^2 - a^2} y + c \)
3. \( z = ax + \frac{k}{a} y + c \)
4. \( \log(az - 1) = x + ay + c \)
5. \( z = a \tan(x + ay + c) \)
6. \( z^2 = (x + ay + c)^2 + a^2 \)
7. \( 2\sqrt{1-a+az} = x + ay + c \)
8. \( z = a(x - y) - \cos x - \cos y + c \)
9. \( z = \frac{1}{3}\left[(x + a)^3 + (y - a)^3\right] + c \)
10. \( z = \frac{2}{3}\left[(x + a)^{3/2} + (y - a)^{3/2}\right] + c \)
11. \( z = ax + by + \left(\frac{a^3}{b^3} + \frac{b^3}{a^3}\right) \)
12. \( z = ax + by - 2\sqrt{ab} \)
13. \( z = ax + by + \sqrt{1 + a^2 + b^2} \)
Non-linear partial differential equations of the first order - Reducible to standard form

I. Equations of the form \( f \left( x^m, y^n \right) = 0 \) and \( f \left( z, x^m, y^n \right) = 0 \)

Case (i): When \( m \neq 1, n \neq 1 \)

Put \( X = x^{1-m} \) and \( Y = y^{1-n} \)

\[
\frac{\partial X}{\partial x} = (1-m)x^{-m}\quad \text{and} \quad \frac{\partial Y}{\partial y} = (1-n)y^{-n}
\]

Now \( p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \frac{\partial X}{\partial x} = \frac{\partial z}{\partial X} (1-m)x^{-m} \) and \( q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \frac{\partial Y}{\partial y} = \frac{\partial z}{\partial Y} (1-n)y^{-n} \)

\[
\Rightarrow p = P(1-m)x^{-m}\quad \text{and} \quad q = Q(1-n)y^{-n}
\]

\[
\Rightarrow x^m p = (1-m)P, \quad \text{where} \quad P = \frac{\partial z}{\partial X} \quad \text{and} \quad y^n q = Q(1-n), \quad \text{where} \quad Q = \frac{\partial z}{\partial Y}
\]

Case (ii): When \( m = 1, n = 1 \)

Put \( X = \log x \) and \( Y = \log y \) so that \( \frac{\partial X}{\partial x} = \frac{1}{x} \) and \( \frac{\partial Y}{\partial y} = \frac{1}{y} \)

Now \( p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \frac{\partial X}{\partial x} = \frac{\partial z}{\partial X} \frac{1}{x} \) and \( q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \frac{\partial Y}{\partial y} = \frac{\partial z}{\partial Y} \frac{1}{y} \)

\[
\Rightarrow p = \frac{1}{X} P\quad \text{and} \quad q = \frac{1}{Y} Q
\]

\[
\Rightarrow xp = P, \quad \text{where} \quad P = \frac{\partial z}{\partial X} \quad \text{and} \quad yq = Q, \quad \text{where} \quad Q = \frac{\partial z}{\partial Y}
\]

II. Equations of the form \( f \left( z^n, z^n \right) = 0 \) and \( f \left( x, z^n \right) = g(y, z^n) \)

To reduce the above equation into standard form, we use the following substitution

\[
Z = \begin{cases} 
\frac{z^{n+1}}{z} & \text{if } n \neq -1 \\
\log z & \text{if } n = -1
\end{cases}
\]

Example 5.1. Solve \( x^2 p^2 + y^2 q^2 = z^2 \)

Solution: Given equation can be written as \( (xp)^2 + (yq)^2 = z^2 \) \hfill (1)

Equation (1) is of the form \( f \left( z, x^m, y^n \right) = 0 \) with \( m = 1 \) and \( n = 1 \).

Put \( X = \log x \) and \( Y = \log y \) so that \( xp = P \) and \( yq = Q \) \hfill (2)

where \( P = \frac{\partial z}{\partial X} \) and \( Q = \frac{\partial z}{\partial Y} \)

From (1) and (2), we get \( P^2 + Q^2 = z^2 \) (Standard form II) \hfill (3)
Assume $Q = aP$ and substitute in (3), we get

$$P^2 + a^2P^2 = z^2 \Rightarrow P = \frac{z}{\sqrt{1+a^2}}$$  \hspace{1cm} (4)

We know that $dz = PdX + QdY \Rightarrow dz = P(dX + adY)$ \hspace{1cm} (\because Q = aP)

$$\Rightarrow dz = \frac{z}{\sqrt{1+a^2}}(dX + adY) \hspace{1cm} [: \text{By (4)}]$$

$$\Rightarrow \frac{\sqrt{1+a^2}}{z}dz = dX + adY$$

Integrating, $\sqrt{1+a^2}\log z = X + aY + c$

or $\sqrt{1+a^2}\log z = \log x + a\log y + c$ \hspace{1cm} (\because X = \log x, Y = \log y)

**Example 5.2.** Solve $z^2(p^2 + q^2) = x^2 + y^2$

**Solution:** Given equation can be written as $(zp)^2 - x^2 = y^2 - (zq)^2$  \hspace{1cm} (1)

Equation (1) is of the form $f(x, z^n p) = g(y, z^n q)$ with $n = 1$

Put $Z = z^{n+1} = z^2$ so that $\frac{\partial Z}{\partial x} = 2z \frac{\partial z}{\partial x} \Rightarrow P = 2zp$ or $zp = \frac{P}{2}$  \hspace{1cm} (2)

Similarly, we can get $zq = \frac{Q}{2}$  \hspace{1cm} (3)

Substituting (2) and (3) in (1), we get $\frac{P^2}{4} - x^2 = y^2 - \frac{Q^2}{4}$ \hspace{1cm} (Standard form III)

Suppose $\frac{P^2}{4} - x^2 = y^2 - \frac{Q^2}{4} = a$

$$\therefore \frac{P^2}{4} - x^2 = a \hspace{1cm} \text{and} \hspace{1cm} y^2 - \frac{Q^2}{4} = a$$

$$\Rightarrow P = 2\sqrt{x^2 + a} \hspace{1cm} \text{and} \hspace{1cm} Q = 2\sqrt{y^2 - a}$$

We know that $dZ = Pdx + Qdy = 2\sqrt{x^2 + a} \ dx + 2\sqrt{y^2 - a} \ dy$

Integrating, $Z = 2\left[\frac{x}{2}\sqrt{x^2 + a} + \frac{a}{2} \log (x + \sqrt{x^2 + a})\right] + 2\left[\frac{y}{2}\sqrt{y^2 - a} - \frac{a}{2} \log (y + \sqrt{y^2 - a})\right] + c$

or $z^2 = x\sqrt{x^2 + a} + y\sqrt{y^2 - a} + a \log \left(\frac{x + \sqrt{x^2 + a}}{y + \sqrt{y^2 - a}}\right) + c$ \hspace{1cm} (\because Z = z^2)

**Exercise 5**

Solve the following

1. $z^2(p^2x^2 + q^2) = 1$
2. $\frac{x^2}{p} + \frac{y^2}{q} = z$
3. $zpy^2 = x(y^2 + z^2q^2)$
4. $z(p^2 - q^2) = x - y$

**Answers:**

1. $z^2\sqrt{1+a^2} = 2(\log x + ay) + c$
2. $z^2 = \frac{2}{3a}(a + 1)(x^3 + ay^3 + c)$
3. \( z^2 = ax^2 + \sqrt{a-1}y^2 + c \)

4. \( z^2 = ax^2 + \sqrt{a-1}y^2 + c \)
Classification of second order partial differential equations

The general linear PDE of the second order in two independent variables \( x, y \) is of the form

\[
A(x, y) \frac{\partial^2 u}{\partial x^2} + B(x, y) \frac{\partial^2 u}{\partial x \partial y} + C(x, y) \frac{\partial^2 u}{\partial y^2} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0
\]  

(1)

Equation (1) is said to be

i) Elliptic, if \( B^2 - 4AC < 0 \),

ii) Parabolic, if \( B^2 - 4AC = 0 \),

iii) Hyperbolic, if \( B^2 - 4AC > 0 \).

Example 6.1. Classify the following partial differential equations:

i) \( \frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial x \partial y} + 4 \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} + 2 \frac{\partial u}{\partial y} = 0 \)

ii) \( (1 + x^2) \frac{\partial^2 u}{\partial x^2} + (5 + 2x^2) \frac{\partial^2 u}{\partial x \partial y} + (4 + x^2) \frac{\partial^2 u}{\partial y^2} = 0 \)

iii) \( x^2 u_{xx} + y^2 u_{yy} - xu_x + yu_y = 0 \)

Solution. i) Comparing this equation with (1) above, we get \( A = 1, B = 4, C = 4 \)

\[
\therefore B^2 - 4AC = (4)^2 - 4(1)(4) = 0
\]

So the given equation is parabolic.

ii) Here \( A = 1 + x^2, B = 5 + 2x^2, C = 4 + x^2 \)

\[
\therefore B^2 - 4AC = (5 + 2x^2)^2 - 4(1 + x^2)(4 + x^2) = 9 > 0
\]

So the given equation is hyperbolic.

iii) Here \( A = x^2, B = 0, C = y^2 \)

\[
\therefore B^2 - 4AC = 0 - 4x^2y^2 = -4x^2y^2 < 0
\]

So the given equation is elliptic.

Exercise 6

Classify the following partial differential equations:

1. \( 3 \frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial x \partial y} + 6 \frac{\partial^2 u}{\partial y^2} - 2 \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - u = 0 \)

2. \( (x + 1)u_{xx} - 2(x + 2)u_{xy} + (x + 3)u_{yy} = 0 \)

3. \( y^2 u_{xx} - 2xyu_{xy} + x^2u_{xy} + 2u_x - 3u = 0 \)

4. \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + (x^2 + 4y^2) \frac{\partial^2 u}{\partial y^2} = 2 \sin xy \)

5. \( x^2 \frac{\partial^2 u}{\partial x^2} + (1 - y^2) \frac{\partial^2 u}{\partial y^2} = 0, -\infty < x < \infty, -1 < y < 1 \)

Answers:

1. Elliptic  
2. Hyperbolic  
3. Parabolic  
4. Elliptic outside the ellipse \( 4x^2 + 16y^2 = 1, \)
Hyperbolic inside the ellipse $4x^2 + 16y^2 = 1$, and Parabolic on the ellipse $4x^2 + 16y^2 = 1$. 
Homogeneous linear equations with constant coefficients-Rules for finding the complimentary function

An equation of the form
\[
\frac{\partial^n x}{\partial x^k} + k_1 \frac{\partial^{n-1} x}{\partial y \partial x^{k-1}} + k_2 \frac{\partial^{n-2} x}{\partial y^2 \partial x^{k-2}} + \ldots + k_n \frac{\partial^n x}{\partial y^n} = F(x, y)
\]  
(1)

where \(k_1, k_2, \ldots, k_n\) are all real constants, is called a homogeneous linear partial differential equation of \(n\)th order with constant coefficients. It is called homogeneous because all terms contain derivatives of the same order.

On writing, \(\frac{\partial'}{\partial x'} = D'\) and \(\frac{\partial'}{\partial y'} = D''\) in (1), we get
\[
\left( D^n + k_1 D^{n-1} + k_2 D^{n-2} + \ldots + k_n \right) z = F(x, y)
\]
or briefly
\[
f(D, D') z = F(x, y)
\]  
(2)

The general solution of (1) consists of two parts, namely: the complimentary function and the particular integral.

**Rules for finding the Complimentary Function (C.F.):**

The complimentary function of (2) is the general solution of \(f(D, D') z = 0\), which must contain \(n\) arbitrary functions.

The algebraic equation
\[
m^n + k_1 m^{n-1} + k_2 m^{n-2} + \ldots + k_n = 0
\]  
(3)

is called the auxiliary equation (A.E) of (2), which is obtained by replacing \(D\) by \(m\) and \(D'\) by 1 in the coefficients of \(z\) in (2).

Suppose \(m_1, m_2, \ldots, m_n\) be \(n\) roots of A.E. (3).

**Theorem.** If \(m\) is a root of A.E then \(z = \phi(y + mx)\) is a linearly independent solution of \(f(D, D') z = 0\).

**Proof.** Since \(m\) is a root of A.E, \((D - mD')\) is a factor of \(f(D, D')\) so that \((D - mD')z = 0\)  
(4)

Equation (4) can be written as \(p - mq = 0\), which is Lagrange’s equation.

The Lagrange’s auxiliary equations are
\[
\frac{dx}{1} = \frac{dy}{-m} = \frac{dy}{0}
\]  
(5)

Taking the first two fractions of (5), we get \(dy + mdx = 0\) so that \(y + mx = c_1\)  
(6)

From the third fraction of (5), we get \(dz = 0\) so that \(z = c_2\)  
(7)

Hence from (6) and (7), the general solution of (4) is given by
\[
z = \phi(y + mx), \text{ where } \phi \text{ is any arbitrary function}
\]
Working rule for finding C.F.:

Step 1. Write the given equation in the standard form

\[ (D^n + k_1D^{n-1} + k_2D^{n-2} + \ldots + k_nD^0)z = F(x, y) \]

Step 2. Replacing \( D \) by \( m \) and \( D' \) by 1 in the coefficients of \( z \), we obtain the auxiliary equation

(A.E) \[ m^n + k_1m^{n-1} + k_2m^{n-2} + \ldots + k_n = 0 \]

Step 3. Solve A.E for \( m \). Two cases will arise.

Case (i) Let \( m = m_1, m_2, \ldots, m_n \) (distinct roots). Then

C.F. = \( \phi_1(y + m_1x) + \phi_2(y + m_2x) + \ldots + \phi_n(y + m_nx) \), where \( \phi_1, \phi_2, \ldots, \phi_n \) are arbitrary functions

In the above case (i), if \( m = \frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n} \), then

C.F. = \( \phi_1(b_1y + a_1x) + \phi_2(b_2y + a_2x) + \ldots + \phi_n(b_ny + a_nx) \)

Case (ii) Let \( m = m' \) (repeated \( k \) times). Then corresponding to these roots

C.F. = \( \phi_1(y + m'x) + x\phi_2(y + m'x) + x^2\phi_3(y + m'x) + \ldots + x^k\phi_{k-1}(y + m'x) \)

In the above case (ii), if \( m = \frac{a}{b} \) (repeated \( k \) times). Then

C.F. = \( \phi_1(by + ax) + x\phi_2(by + ax) + x^2\phi_3(by + ax) + \ldots + x^k\phi_{k-1}(by + ax) \)

Example 7.1. Solve \( 2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x\partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0 \)

Solution. Given equation can be written in operator form as \( (2D^2 + 5DD' + 2D'^2)z = 0 \)

Its auxiliary equation is \( 2m^2 + 5m + 2 = 0 \)

\[ \Rightarrow (m + 2)(2m + 1) = 0 \Rightarrow m = -2, -\frac{1}{2} \]

Hence the general solution is \( z = f_1(y - 2x) + f_2(2y - x) \)

Example 7.2. Solve \( 4r + 12s + 9t = 0 \)

Solution. Since \( r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x\partial y}, \) and \( t = \frac{\partial^2 z}{\partial y^2} \), given equation can be written as

\[ 4\frac{\partial^2 z}{\partial x^2} + 12\frac{\partial^2 z}{\partial x\partial y} + 9\frac{\partial^2 z}{\partial y^2} = 0 \]

Its operator form is \( (4D^2 + 12DD' + 9D'^2)z = 0 \)

The auxiliary equation is \( 4m^2 + 12m + 9 = 0 \)

\[ \Rightarrow (2m + 3)^2 = 0 \Rightarrow m = -\frac{3}{2}, -\frac{3}{2} \]

Hence the general solution is \( z = f_1(2y - 3x) + xf_2(2y - 3x) \)

Example 7.3. Solve \( \frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0 \)
Solution. Given equation can be written in operator form as \((D^4 - D'^4)z = 0\)

Its auxiliary equation is \(m^4 - 1 = 0\)

\[
\Rightarrow (m^2 - 1)(m^2 + 1) = 0
\]

\[
\Rightarrow m = -1, 1, i, -i
\]

Hence the general solution is \(z = f_1(y + x) + f_2(y - x) + f_3(y + ix) + f_4(y - ix)\)

Example 7.4. Solve \((D^4 + D'^4)z = 0\)

Solution. The auxiliary equation is \(m^4 + 1 = 0 \Rightarrow (m^2 + 1)^2 - 2m^2 = 0\)

\[
\Rightarrow (m^2 + 1)^2 - (m\sqrt{2})^2 = 0 \Rightarrow (m^2 + m\sqrt{2} + 1)(m^2 - m\sqrt{2} + 1) = 0
\]

so that \(m^2 + m\sqrt{2} + 1 = 0\) or \(m^2 - m\sqrt{2} + 1 = 0 \Rightarrow m = \frac{-1+i\sqrt{2}}{2}, \frac{1+i\sqrt{2}}{2}\)

Let \(z_1 = \frac{-1+i\sqrt{2}}{2}\) and \(z_2 = \frac{1+i\sqrt{2}}{2}\), then \(\bar{z}_1 = \frac{-1-i\sqrt{2}}{2}\) and \(\bar{z}_2 = \frac{1-i\sqrt{2}}{2}\).

Hence the general solution is \(z = f_1(y + z_1x) + f_2(y + \bar{z}_1x) + f_3(y + z_2x) + f_4(y + \bar{z}_2x)\)

Exercise 7

Solve the following

1. \(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 6 \frac{\partial^2 z}{\partial y^2} = 0\)

2. \(25r - 40s + 16t = 0\)

3. \(\frac{\partial^3 z}{\partial x^3} - 4 \frac{\partial^3 z}{\partial x^2 \partial y} + 4 \frac{\partial^3 z}{\partial x \partial y^2} = 0\)

4. \((D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0\)

5. \((D^4 - 2D^3D' + 2DD'^3 - D'^4)z = 0\)

6. \((D^4 + D'^2 - 2D^2D'^2)z = 0\)

Answers:

1. \(z = f_1(y - 2x) + f_2(y + 3x)\)

2. \(z = f_1(5y + 4x) + xf_2(5y + 4x)\)

3. \(z = f_1(y) + f_2(y + 2x) + xf_3(y + 2x)\)

4. \(z = f_1(y + x) + f_2(y + 2x) + f_3(y + 3x)\)

5. \(z = f_1(y + x) + xf_2(y + x) + x^2f_3(y + x) + f_4(y - x)\)

6. \(z = f_1(y + x) + xf_2(y + x) + f_3(y - x) + xf_4(y - x)\)
A4011 - PARTIAL DIFFERENTIAL EQUATIONS AND COMPLEX VARIABLES

Handout #8

Homogeneous linear equations with constant coefficients - Rules for finding the particular integral

Consider a homogeneous linear PDE of \( n \)th order with constant coefficients

\[
f(D,D') z = F(x,y) \ldots (1)
\]

The particular integral of (1) is given by

\[
P.I = \frac{1}{f(D,D')} F(x,y)
\]

Rules for finding the particular integral (P.I):

**Case 1:** When \( F(x,y) = \phi(ax+by) \), where \( \phi \) is an elementary function of \( ax+by \)

\[
P.I = \frac{1}{f(D,D')} \phi(ax+by)
\]

\[
= \frac{1}{f(a,b)} \int \ldots \int \phi(v) dv \ldots dv, \text{ where } v = ax+by
\]

Result-I:

\[
\frac{1}{(bD-aD')^k} \phi(ax+by) = \frac{x^k}{k! \cdot b^k} \phi(ax+by)
\]

**Case 2:** If \( f(a,b) = 0 \) then

\[
f(D,D') = (bD-aD')^r (cD+dD')^s
\]

\[
P.I = \frac{1}{(bD-aD')^r (cD+dD')^s} \phi(ax+by)
\]

\[
= \frac{1}{(bD-aD')^r} \left[ \frac{1}{(cD+dD')^s} \phi(ax+by) \right]
\]

\[
= \frac{1}{(bD-aD')^r} \left[ \frac{1}{(ca+db)^s} \int \ldots \int \phi(v) dv \ldots dv, \text{ where } v = ax+by \right]
\]

\[
= \frac{1}{(ca+db)^s} \left[ \frac{1}{(bD-aD')^r} \psi(ax+by) \right]
\]

\[
= \frac{1}{(ca+db)^s} \left[ \frac{x^r}{r! \cdot b^r} \psi(ax+by) \right]
\]

\[
= \frac{1}{(bD-aD')^k} \phi(ax+by) = \frac{x^k}{k! \cdot b^k} \phi(ax+by)
\]

**Case 3:** When \( F(x,y) = x^m y^n \), where \( m, n \) are positive constants

\[
P.I = \frac{1}{f(D,D')} x^m y^n = \left[ 1 \pm \phi\left( \frac{D}{D'} \right) \right]^{-1} x^m y^n
\]
Write \( \frac{1}{f(D, D')} \) in the form of \( \left[ 1 \pm \phi \left( \frac{D'}{D} \right) \right]^{-1} \) by taking highest degree term of \( D \) as common from \( f(D, D') \), expand it in ascending powers of \( \left( \frac{D'}{D} \right) \) and then operate on \( x^m y^n \) term by term.

**Result-II:** 
\[
\frac{1}{D - mD'} F(x, y) = \left[ \int F(x, c - mx) \, dx \right]_{c=y+mx}
\]

**Example 8.1.** Solve \( \frac{\partial^3 z}{\partial x^3} - 3 \frac{\partial^3 z}{\partial x^2 \partial y} + 4 \frac{\partial^3 z}{\partial y^3} = e^{x+2y} \)

**Solution.** Given equation in operator form is \( \left( D^3 - 3D^2 D' + 4D'^3 \right) z = e^{x+2y} \)

Its auxiliary equation is \( m^3 - 3m^2 + 4 = 0 \Rightarrow (m+1)(m-2)^2 = 0 \Rightarrow m = -1, 2, 2 \)

\[
\therefore \text{C.F} = f_1(y-x) + f_2(y+2x) + xf_3(y+2x)
\]

**P.I** = \[
\frac{1}{D^3 - 3D^2 D' + 4D'^3} e^{x+2y}
\]

\[
= \frac{1}{1-3(2)+4(2)} \int \int \int e^v dv dv dv, \text{ where } v = x + 2y
\]

\[
= \frac{1}{27} e^v
\]

\[
= \frac{1}{27} e^{x+2y}
\]

Hence the general solution is \( z = \text{C.F} + \text{P.I} = f_1(y-x) + f_2(y+2x) + xf_3(y+2x) + \frac{1}{27} e^{x+2y} \)

**Example 8.2.** Solve \( \frac{\partial^3 z}{\partial x^3} - 2 \frac{\partial^3 z}{\partial x^2 \partial y} - 2 \frac{\partial^3 z}{\partial y^3} = e^{2x+y} \)

**Solution.** Given equation in operator form is \( \left( D^3 - 4D^2 D' + 5DD'^2 - 2D'^3 \right) z = e^{2x+y} \)

Its auxiliary equation is \( m^3 - 4m^2 + 5m - 2 = 0 \Rightarrow (m-1)^2(m-2) = 0 \Rightarrow m = 1, 1, 2 \)

\[
\therefore \text{C.F} = f_1(y+x) + xf_2(y+x) + f_3(y+2x)
\]

**P.I** = \[
\frac{1}{D^3 - 4D^2 D' + 5DD'^2 - 2D'^3} e^{2x+y} \quad \text{(Put } D = 2, D' = 1, \text{ we get } f(D, D') = 0 )
\]

\[
= \frac{1}{D-2D'} \left( \frac{1}{(D-D')} e^{2x+y} \right)
\]

\[
= \frac{1}{D-2D'} \left( \frac{1}{(2-1)} \int e^v dv dv \right), \text{ where } v = 2x + y
\]

\[
= \frac{1}{D-2D'} e^{2x+y}
\]

\[
= \frac{1}{D-2D'} e^{2x+y} = xe^{2x+y}
\]

Hence the general solution is \( z = \text{C.F} + \text{P.I} = f_1(y+x) + xf_2(y+x) + f_3(y+2x) + xe^{2x+y} \)

**Example 8.3.** Solve \( (D^3 - 4D^2 D' + 4DD'^2) z = 2 \sin(3x + 2y) \)
Solution. The auxiliary equation is \( m^3 - 4m^2 + 4m = 0 \Rightarrow m(m-2)^2 = 0 \Rightarrow m = 0, 2, 2 \)

\[ \therefore C.F = f_1(y) + f_2(y + 2x) + xf_3(y + 2x) \]

\[ P.I = \frac{1}{D^3 - 4D^2 D' + 4DD'^2} \sin(3x + 2y) \]

\[ = \frac{1}{(3)^3 - 4(3)^2 (2) + 4(3)(2)^2} \int \int \sin(3x + 2y) \]

\[ = \frac{2}{3} \cos(3x + 2y) \]

Hence the general solution is \( z = C.F + P.I = f_1(y) + f_2(y + 2x) + xf_3(y + 2x) + \frac{2}{3} \cos(3x + 2y) \)

**Example 8.4.** Solve \( (D^2 - 6DD' + 9D'^2)z = \tan(y + 3x) \)

Solution. The auxiliary equation is \( m^2 - 6m + 9 = 0 \Rightarrow (m-3)^2 = 0 \Rightarrow m = 3, 3 \)

\[ \therefore C.F = f_1(y + 3x) + xf_2(y + 3x) \]

\[ P.I = \frac{1}{D^2 - 6DD' + 9D'^2} \tan(y + 3x) \]

\[ = \frac{1}{(D-3)^2} \tan(y + 3x) \frac{1}{(bD - aD') \phi(ax + by) - \frac{x^k}{k!} b^k \phi(ax + by)} \]

\[ = \frac{x^2}{2!} \tan(y + 3x) = \frac{x^2}{3} \tan(y + 3x) \]

Hence the general solution is \( z = C.F + P.I = f_1(y + 3x) + xf_2(y + 3x) + \frac{x^2}{3} \tan(y + 3x) \)

**Example 8.5.** Solve \( (D^2 - DD' - 2D'^2)z = (y - 1)e^x \)

Solution. The auxiliary equation is \( m^2 - m + 2 = 0 \Rightarrow (m+1)(m-2) = 0 \Rightarrow m = -1, 2 \)

\[ \therefore C.F = f_1(y-x) + f_2(y + 2x) \]

\[ P.I = \frac{1}{D^2 - DD' - 2D'^2} (y-1)e^x \]

\[ = \frac{1}{(D+D')(D-2D')} (y-1)e^x \]

\[ = \frac{1}{(D+D')} \left[ \frac{1}{D-2D'} (y-1)e^x \right] \]

\[ = \frac{1}{(D+D')} \left[ \int e^x (c-2x-1) dx \right]_{c=y+2x} = \frac{1}{(D+D')} \left[ \int F(x, c-mx) dx \right]_{c=y+mx} \]

\[ = \frac{1}{(D+D')} \left[ e^x [(c-2x-1) - (-2)] \right]_{c=y+2x} \]

\[ = \frac{1}{(D+D')} e^x (y+1) \]

\[ \therefore \frac{1}{D+D'} F(x, y) = \int F(x, c+mx) dx \]

\[ \left. \right|_{c=y-mx} \]
\[
\int e^{x}(c + x + 1)dx \bigg|_{c=y-x} = \int e^{x}(c + x + 1 - 1)dx \bigg|_{c=y-x} = ye^x
\]

Hence the general solution is \( z = C.F + P.I = f_1(y - x) + f_2(y + 2x) + ye^x \)

**Example 8.6.** Solve \( \frac{\partial^2 z}{\partial x^2} + 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = x^2 + xy + y^2 \)

**Solution.** Given equation in operator form is \( (D^2 + 2DD' + D'^2)z = x^2 + xy + y^2 \)

Its auxiliary equation is \( m^2 + 2m + 1 = 0 \Rightarrow (m + 1)^2 = 0 \Rightarrow m = -1, -1 \)

\( \therefore \) C.F = \( f_1(y - x) + xf_2(y - x) \)

\( \text{P.I} = \frac{1}{D^2 + 2DD' + D'^2} (x^2 + xy + y^2) = \frac{1}{(D + D')^2} (x^2 + xy + y^2) \)

\[= \frac{1}{D^2} \left( 1 + \frac{D'}{D} \right)^2 (x^2 + xy + y^2) \]

\[= \frac{1}{D^2} \left( 1 - 2 \frac{D'}{D} + \frac{3D'^2}{D^2} - \ldots \right) (x^2 + xy + y^2) \]

\[= \frac{1}{D^2} \left[ x^2 + xy + y^2 - \frac{2}{D} (x + 2y) + \frac{3}{D^2} (2) \right] \]

\[= \frac{1}{D^2} \left[ x^2 + xy + y^2 - 2 \left( \frac{x^2}{2} + 2xy \right) + 6 \cdot \frac{x^2}{2} \right] \]

\[= \frac{1}{D^2} \left( 3x^2 - 3xy + y^2 \right) \]

\[= 3 \frac{1}{D^2} (x^2) - 3y \frac{1}{D^2} (x) + y^2 \frac{1}{D^2} (1) \]

\[= 3 \left( \frac{x^4}{3} \right) - 3y \left( \frac{x^3}{2} \right) + y^2 \left( \frac{x^2}{2} \right) \]

\[= \frac{1}{4} (x^4 - 2x^3y + 2x^2y^2) \]

Hence the general solution is \( z = C.F + P.I = f_1(y - x) + xf_2(y - x) + \frac{1}{4} (x^4 - 2x^3y + 2x^2y^2) \)

**Exercise 8**

Solve the following:

1. \( r - 4s + 4t = e^{2x+y} \)
2. \( (D^2 - 2DD' + D'^2)z = e^{x+y} \)
3. \( \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6 \frac{\partial^2 z}{\partial y^2} = \cos(2x + y) \)
4. \( (D^3 - 7DD'^2 - 6D^3)z = \cos(2x + y) \)
5. \( \frac{\partial^2 z}{\partial x^2} - 3 \frac{\partial^2 z}{\partial x \partial y} + 2 \frac{\partial^2 z}{\partial y^2} = e^{2x-y} + e^{x+y} + \cos(x + 2y) \)
6. \( \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \cos x \cos 2y \)
7. \[ 4 \frac{\partial^2 z}{\partial x^2} - 4 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 16 \log(x + 2y) \]

8. \[ (D^2 + 3DD' + 2D'^2)z = 24xy \]

9. \[ \frac{\partial^3 z}{\partial x^3} - 2 \frac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x} + 3x^2y \]

10. \[ (D^2 - 6DD' + 9D'^2)z = 12x^2 + 36xy \]

11. \[ r + s - 6t = y \cos x \]

12. \[ (D^2 + DD' - 6D'^2)z = y \sin x \]

13. \[ (D^3 + D^2D'^2 - DD'^2 - D'^3)z = e^x \cos 2y \]

14. \[ (D^2 + 2DD' + D'^2)z = 2 \cos x - x \sin y \]

Answers:

1. \[ z = f_1(y + 2x) + xf_2(y + 2x) + \frac{x^2}{2} e^{2x+y} \]

2. \[ z = f_1(y + x) + xf_2(y + x) + \frac{x^2}{2} e^{x+y} \]

3. \[ z = f_1(y + 2x) + f_2(y - 3x) + \frac{x}{3} \sin(2x + y) \]

4. \[ z = f_1(y - x) + f_2(y - 2x) + f_3(y + 3x) + \frac{1}{12} \sin(2x + y) \]

5. \[ z = f_1(y + x) + f_2(y + 2x) + \frac{1}{12} e^{2x-y} - xe^{x+y} - \frac{1}{3} \cos(x + 2y) \]

6. \[ z = f_1(y) + f_2(y + x) + \frac{1}{6} [3 \cos(x + 2y) - \cos(x - 2y)] \]

7. \[ z = f_1(2y + x) + xf_2(2y + x) + 2x^2 \log(x + 2y) \]

8. \[ z = f_1(y - x) + f_2(y - 2x) + 4x^3 - 3x^4 \]

9. \[ z = f_1(y) + xf_2(y) + f_3(y + 2x) + \frac{1}{60} \left(15e^{2x} + 3x^5y + x^6\right) \]

10. \[ z = f_1(y + 3x) + xf_2(y + 3x) + 10x^3 + 6x^3y \]

11. \[ z = f_1(y + 2x) + f_2(y - 3x) - y \cos x + \sin x \]

12. \[ z = f_1(y + 2x) + f_2(y - 3x) - y \sin x + \cos x \]

13. \[ z = f_1(y + x) + f_2(y - x) + xf_3(y - x) + \frac{e^x}{25} (\cos 2y + 2 \sin 2y) \]

14. \[ z = f_1(y - x) + xf_3(y - x) + x \sin y \]
Non-homogeneous linear equations with constant coefficients

In the equation \( f(D, D') z = F(x, y) \) (1)

if the polynomial expression \( f(D, D') \) is not homogeneous, then (1) is said to be a non-homogeneous linear partial differential equation.

**Rules for finding the Complementary Function (C.F.):**

To find the C.F. we resolve \( f(D, D') \) into linear factors of the form \( aD + bD' + c \).

Now consider the equation \( (aD + bD' + c) z = 0 \),

or \( ap + bq = -cz \) (2)

The Lagrange’s auxiliary equations of (2) are \( \frac{dx}{a} = \frac{dy}{b} = \frac{dz}{-cz} \) (3)

Taking the first two fractions of (3), we get \( ady - bdx = 0 \) so that \( ay - bx = c_1 \)

**Case (i): When \( a \neq 0 \)**

Taking the first and third fractions of (3), we get \( -\frac{c}{a} dx = \frac{dz}{cz} \)

Integrating, we get \( \log z = -\frac{c}{a} x + \log c_2 \) so that \( z = c_2 e^{-\frac{c}{a} x} \) (4)

Taking \( c_2 = \phi(c_1) \) in (4), we get \( z = \phi(c_1) e^{-\frac{c}{a} x} = e^{-\frac{c}{a} x} \phi(ay - bx) \)

The general solution of \( (aD + bD' + c) z = 0 \) is \( z = e^{-\frac{c}{a} x} \phi(ay - bx), \quad a \neq 0 \)

**Case (ii): When \( a = 0 \)**

Taking the second and third fractions of (3), we get \( -\frac{c}{b} dy = \frac{dz}{cz} \)

Integrating, we get \( \log z = -\frac{c}{b} y + \log c_3 \) so that \( z = c_3 e^{-\frac{c}{b} y} \) (5)

Taking \( c_3 = \phi(c_1) \) in (5), we get \( z = \phi(c_1) e^{-\frac{c}{b} y} = e^{-\frac{c}{b} y} \phi(-bx) \)

The general solution of \( (bD' + c) z = 0 \) is \( z = e^{-\frac{c}{b} y} \phi(bx), \) where \( \phi(-bx) = \psi(bx) \)

**Rules for finding the Particular Integral (P.I.):**

**Case (i): When \( F(x, y) = e^{ax+by} \)**

\[
P.I. = \frac{1}{f(D, D')} e^{ax+by} = \frac{1}{f(a, b)} e^{ax+by}, \text{ provided } f(a, b) \neq 0
\]
Failure case: If \( f(a,b) = 0 \), then
\[
P.I = x \cdot \frac{1}{\partial D \{ f(D,D') \}} e^{ax+by} = x \cdot \frac{1}{g(D,D')} e^{ax+by}
\]
\[
= x \cdot \frac{1}{g(a,b)} e^{ax+by}, \text{ provided } g(a,b) \neq 0
\]

Case (ii): When \( F(x,y) = \sin(ax+by) \) or \( \cos(ax+by) \)

Put \( D^2 = -a^2, DD' = -ab, D'^2 = -b^2 \), we get
\[
P.I = \frac{1}{f(D^2, DD', D'^2)} \sin(ax+by) \text{ or } \cos(ax+by)
\]
\[
= \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax+by) \text{ or } \cos(ax+by), \text{ provided } f(-a^2, -ab, -b^2) \neq 0
\]

If \( f(-a^2, -ab, -b^2) = 0 \), we apply the failure case stated in case (i).

Example 9.1. Solve \( (D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{2x+3y} \)

Solution. Given equation can be written as \( (D + D')(D - 2D' + 2)z = e^{2x+3y} \)

The part of the C.F corresponding to the factor \( D + D' \) is \( z = f_1(y-x) \) \( (: a = 1, b = 1, c = 0) \)

The part of the C.F corresponding to \( D - 2D' + 2 \) is \( z = e^{-2x} f_2(y + 2x) \) \( (: a = 1, b = -2, c = 2) \)

\[
\therefore \text{C.F} = f_1(y-x) + e^{-2x} f_2(y + 2x)
\]

\[
P.I = \frac{1}{D^2 - DD' - 2D'^2 + 2D + 2D'} e^{2x+3y}
\]

Put \( D = 2, D' = 3 \), we get
\[
= \frac{1}{2^2 - (2 \times 3) - 2(3)^2 + 2(2) + 2(3)} e^{2x+3y}
\]
\[
= -\frac{1}{10} e^{2x+3y}
\]

Hence the general solution is \( z = \text{C.F} + \text{P.I} = f_1(y-x) + e^{-2x} f_2(y + 2x) - \frac{1}{10} e^{2x+3y} \)

Example 9.2. Solve \( (D^2 + DD' + D' - 1)z = \sin(x + 2y) \)

Solution. Given equation can be written as \( (D + 1)(D + D' - 1)z = \sin(x + 2y) \)

The part of the C.F corresponding to the factor \( D + 1 \) is \( z = e^{-x} f_1(y) \) \( (: a = 1, b = 0, c = 1) \)

The part of the C.F corresponding to the factor \( D + D' - 1 \) is

\[
z = e^{-x} f_2(y-x) \) \( (: a = 1, b = 1, c = -1) \)
\]

\[
\therefore \text{C.F} = e^{-x} f_1(y) + e^{x} f_2(y-x)
\]
P.I. = \frac{1}{D^2 + DD' + D' - 1} \sin(x + 2y)

Put \( D^2 = -1^2, DD' = -1 \cdot 2, D'^2 = -2^2 \), we get

\begin{align*}
&= \frac{1}{-1 - 2 + D' - 1} \sin(x + 2y) = \frac{1}{D' - 4} \sin(x + 2y) \\
&= \frac{(D' + 4)}{D'^2 - 16} \sin(x + 2y) = \frac{(D' + 4)}{-2^2 - 16} \sin(x + 2y) \\
&= \frac{-1}{20} \left[ D' \{\sin(x + 2y)\} + 4 \sin(x + 2y) \right] \\
&= \frac{-1}{10} \left[ \cos(x + 2y) + 2 \sin(x + 2y) \right]
\end{align*}

Hence the general solution is \( z = \text{C.F} + \text{P.I} \)

\[ z = e^{-x} f_1(y) + e^x f_2(y - x) - \frac{1}{10} \left[ \cos(x + 2y) + 2 \sin(x + 2y) \right] \]

**Example 9.3.** Solve \( (D^2 - DD' + D' - 1)z = e^y + xy \)

**Solution.** Given equation can be written as \( (D - 1)(D - D' + 1)z = e^y + xy \)

\[ \therefore \text{C.F} = e^x f_1(y) + e^{-x} f_2(y + x) \]

\[ \therefore \text{P.I} = \frac{1}{D^2 - DD' + D' - 1} (e^y + xy) \]

\[ = \frac{1}{D^2 - DD' + D' - 1} e^y + \frac{1}{D^2 - DD' + D' - 1} xy \]

\[ = \text{P.I}_1 + \text{P.I}_2 \]

\[ \text{P.I}_1 = \frac{1}{D^2 - DD' + D' - 1} e^{0, x+1, y} \]

Put \( D = 0, D' = 1 \), we get \( f(D, D') = 0 \)

\[ = x \cdot \frac{1}{\frac{\partial}{\partial D} \left( D^2 - DD' + D' - 1 \right)} e^{0, x+1, y} = x \cdot \frac{1}{2D - D'} e^{0, x+1, y} \]

\[ = x \cdot \frac{1}{2 \cdot 0 - 1} e^y = -xe^y \]

\[ \text{P.I}_2 = \frac{1}{D^2 - DD' + D' - 1} xy = \frac{1}{(D - 1)(D - D' + 1)} xy = -(1 - D)^{-1} \left[ 1 + (D - D') \right]^{-1} xy \]

\[ = -(1 + D + D^2 + .......) \left[ 1 - (D - D') + (D - D')^2 - ....... \right] xy \]

\[ = -(1 + D + D^2 + .......) \left[ (1 - D + D' - 2DD' + ..... )xy \right] \]

\[ = -(1 + D + D^2 + .......) \left[ xy - D(xy) + D'(xy) - 2DD'(xy) \right] \]

\[ = -(1 + D + D^2 + .......) (xy - y + x - 2) = - \left[ (xy - y + x - 2) + D(xy - y + x - 2) \right] \]


\[ = -xy - x + 1 \]

\[ \therefore \text{ P.I} = \text{P.I}_1 + \text{P.I}_2 = -xe^y - xy - x + 1 \]

Hence the general solution is \[ z = C.F + \text{P.I} = e^x f_1(y) + e^{-x} f_2(y + x) - xe^y - xy - x + 1 \]

**Exercise 9**

Solve the following

1. \((D - D' - 1)(D - D' - 2)z = e^{2x-y}\)

2. \(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y} - z = e^{-x}\)

3. \((2DD' + D'^2 - 3D')z = 3\cos(3x - 2y)\)

4. \(\left(D^2 + 2DD' + D'^2 - 2D - 2D'\right)z = \sin(x + 2y)\)

5. \((D + D' - 1)(D + 2D' - 3)z = 4 + 3x + 6y\)

6. \((D^2 - DD' - 2D)z = \sin(3x + 4y) + x^2y\)

7. \((D^2 - D'^2 - 3D + 3D')z = xy + e^{x+2y}\)

**Answers:**

1. \[ z = e^x f_1(y + x) + e^{2x} f_2(y + x) + \frac{1}{2} e^{2x-y} \]

2. \[ z = e^{-x} f_1(y) + e^x f_2(y - x) - \frac{1}{2} e^{-x} \]

3. \[ z = f_1(x) + e^{3x} f_2(2y - x) + \frac{3}{50} \left[4 \cos(3x - 2y) + 3 \sin(3x - 2y)\right] \]

4. \[ z = f_1(y - x) + e^{2x} f_2(y - x) + \frac{1}{39} \left[2 \cos(x + 2y) - 3 \sin(x + 2y)\right] \]

5. \[ z = e^x f_1(y - x) + e^{3x} f_2(y - 2x) + x + 2y + 6 \]

6. \[ z = f_1(y) + e^{2x} f_2(y + x) + \frac{1}{15} \left[\sin(3x + 4y) + 2 \cos(3x + 4y)\right] \\
- \frac{1}{24} (4x^3y + 6x^2y - 2x^3 + 6xy - 6x^2 - 9x) \]

7. \[ z = f_1(y + x) + e^{3x} f_2(y - x) - \frac{1}{54} (3x^3 + 9x^2y + 6xy + 6x^2 + 4x) - xe^{x+2y} \]
**A4011 - PARTIAL DIFFERENTIAL EQUATIONS AND COMPLEX VARIABLES**

**Handout # 10**

**Method of separation of variables**

Method of separation of variables is a powerful technique to solve partial differential equations. Consider a PDE in the function $u$ of two independent variables $x, y$ of the form

$$ F\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}, \ldots \right) = 0 \quad (1) $$

Assume that the required solution be $u(x, y) = X(x)Y(y) \ldots (2)$, where $X(x)$ is a function of $x$ alone and $Y(y)$ is a function of $y$ alone. Then substitution of $u$ from (2) and its derivatives reduces equation (1) to the form

$$ f\left(X, X', X'', \ldots \right) = g\left(Y, Y', Y'', \ldots \right) \quad (3) $$

which is separable in $X$ and $Y$. Since the L.H.S of (3) is a function of $x$ alone and the R.H.S of (3) is a function of $y$ alone, then (3) must be equal to a common constant say $k$. Thus (3) reduces to

$$ f\left(X, X', X'', \ldots \right) = k \quad (4) \quad \text{and} \quad g\left(Y, Y', Y'', \ldots \right) = k \quad (5) $$

Solve (4) and (5) for $X,Y$ and substitute in (2) to get the required solution.

**Example 10.1.** Solve $2 \frac{\partial^2 z}{\partial x^2} + 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$ by the method of separation of variables.

**Solution.** Assume that $z(x, y) = X(x)Y(y) \ldots (1)$

Differentiating (1) w.r.t. $x$ and $y$, we get $\frac{\partial z}{\partial x} = X'Y, \frac{\partial z}{\partial y} = XY', \frac{\partial^2 z}{\partial x \partial y} = X''Y$

Substituting these values in the given equation, we get

$$ X''Y - 2X'Y + XY' = 0, \quad \text{where} \quad X' = \frac{dX}{dx} \quad \text{and} \quad Y' = \frac{dY}{dy} $$

Separating the variables, we get

$$ \frac{X''-2X'}{X} = \frac{-Y'}{Y} = k = \text{constant} $$

$$ \therefore \quad \frac{X''-2X'}{X} = k \quad \text{and} \quad \frac{Y'}{Y} = -k $$

$$ \Rightarrow \quad X'' - 2X' - kX = 0 \quad (2) \quad \text{and} \quad Y' + kY = 0 \quad (3) $$

Equation (2) can be written as $d^2X/dx^2 - 2 dX/dx - kX = 0$ or $(D^2 - 2D - k)X = 0$

The auxiliary equation is $m^2 - 2m - k = 0 \Rightarrow m = 1 \pm \sqrt{1+k}$

Its solution is $X = c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x}$

Equation (3) can be written as $dY/dy + kY = 0$ or $dY/Y = -kdY$
Integrating, we get \( \log Y = -ky + \log c_3 \) or \( Y = c_3 e^{-ky} \)

Substituting these values in (1), we get

\[
z = \left[ c_1 e^{(1+\sqrt{k})x} + c_2 e^{(1-\sqrt{k})x} \right] c_3 e^{-ky}
\]

or

\[
z = Ae^{(1+\sqrt{k})x-ky} + Be^{(1-\sqrt{k})x-ky}, \text{ where } c_1c_3 = A, \ c_2c_3 = B
\]

**Example 10.2.** Use method of separation of variables to solve \( \frac{\partial u}{\partial x} = 4 \frac{\partial u}{\partial y} \), given that \( u(0, y) = 8e^{-3y} \)

**Solution.** Assume that \( u(x, y) = X(x)Y(y) \ldots (1) \)

Differentiating (1) w.r.t. \( x \) and \( y \), we get \( \frac{\partial u}{\partial x} = X'Y, \ \frac{\partial u}{\partial y} = XY' \)

Substituting these values in the given equation, we get

\( X'Y = 4XY' \)

Separating the variables, we get

\[
\frac{X'}{X} = \frac{4Y'}{Y} = k = \text{constant}
\]

\[
\therefore \quad \frac{X'}{X} = k \quad \text{and} \quad \frac{4Y'}{Y} = k
\]

\( \Rightarrow X' - kX = 0 \ldots (2) \quad \text{and} \quad 4Y' - kY = 0 \ldots (3) \)

Equation (2) can be written as \( \frac{dX}{dx} - kX = 0 \) or \( \frac{dX}{X} = k \ dx \)

Integrating, we get \( \log X = kx + \log c_1 \) or \( X = c_1 e^{kx} \)

Equation (3) can be written as \( \frac{4dY}{dy} - kY = 0 \) or \( \frac{dY}{Y} = \frac{k}{4} dy \)

Integrating, we get \( \log Y = \frac{k}{4} y + \log c_2 \) or \( Y = c_2 e^{\frac{ky}{4}} \)

Thus \( u(x, y) = c_1 e^{kx} \cdot c_2 e^{\frac{ky}{4}} = Ae^{\frac{ky}{4}} \ldots (4) \), where \( c_1c_2 = A \)

Given that \( u(0, y) = 8e^{-3y} \Rightarrow Ae^{\frac{ky}{4}} = 8e^{-3y} \Rightarrow A = 8 \) and \( \frac{k}{4} = -3 \Rightarrow k = -12 \)

Substituting these values in (4), we get \( u(x, y) = 8e^{-3(4x+y)} \)

**Example 10.3.** Solve \( py^3 + qx^2 = 0 \) by the method of separation of variables

**Solution.** Given equation can be written as \( y^3 \frac{\partial z}{\partial x} + x^2 \frac{\partial z}{\partial y} = 0 \)

Assume that \( z(x, y) = X(x)Y(y) \ldots (1) \)

Differentiating (1) w.r.t. \( x \) and \( y \), we get \( \frac{\partial z}{\partial x} = X'Y, \ \frac{\partial z}{\partial y} = XY' \)

Substituting these values in the given equation, we get
\[ y^3 X'Y + x^2 XY' = 0, \]  
where \( X' = \frac{dX}{dx} \) and \( Y' = \frac{dY}{dy} \)

Separating the variables, we get

\[ \frac{X'}{x^2 X} = -\frac{Y'}{y^3 Y} = k = \text{constant} \]

\[ \therefore \frac{X'}{x^2 X} = k \quad \text{and} \quad \frac{Y'}{y^3 Y} = -k \]

\[ \Rightarrow X' - kx^2 X = 0 \quad \ldots \ (2) \quad \text{and} \quad Y' + ky^3 Y = 0 \quad \ldots \ (3) \]

Equation (2) can be written as \( \frac{dX}{dx} - kx^2 X = 0 \) or \( \frac{dX}{X} = kx^2 \ dx \)

Integrating, we get \( \log X = \frac{k}{3} x^3 + \log c_1 \) or \( X = c_1 e^{\frac{kx^3}{3}} \)

Equation (3) can be written as \( \frac{dY}{dy} + ky^3 Y = 0 \) or \( \frac{dY}{Y} = -ky^3 \ dy \)

Integrating, we get \( \log Y = -\frac{k}{4} y^4 + \log c_2 \) or \( Y = c_2 e^{-\frac{kx^4}{4}} \)

Substituting these values in (1), we get

\[ z(x, y) = c_1 e^{\frac{kx^3}{3}} \cdot c_2 e^{-\frac{kx^4}{4}} \]

or

\[ z(x, y) = Ae^{\left(\frac{kx^3}{3} - \frac{kx^4}{4}\right)} \], where \( c_1c_2 = A \)

**Example 10.4.** Find a solution of \( \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u \) in the form \( u = X(x)Y(y) \). Solve the equation subject to the conditions \( u = 0 \) and \( \frac{\partial u}{\partial x} = 1 + e^{xy} \), when \( x = 0 \) for all values of \( y \).

**Solution.** Assume that \( u(x, y) = X(x)Y(y) \) \ldots (1)

Differentiating (1) w.r.t. \( x \) and \( y \), we get \( \frac{\partial u}{\partial x} = X'Y, \frac{\partial u}{\partial y} = XY' \) and \( \frac{\partial^2 u}{\partial x^2} = X''Y \)

Substituting these values in the given equation, we get

\[ X''Y = XY' + 2XY \]

Separating the variables, we get

\[ \frac{X''}{X} = \frac{Y'+2Y}{Y} = k = \text{constant} \]

\[ \therefore \frac{X''}{X} = k \quad \text{and} \quad \frac{Y'+2Y}{Y} = k \]

\[ \Rightarrow X'' - kX = 0 \quad \ldots \ (2) \quad \text{and} \quad Y' = (k-2)Y \quad \ldots \ (3) \]

Equation (2) can be written as \( \frac{d^2 X}{dx^2} - kX = 0 \) or \( (D^2 - k)X = 0 \)

The auxiliary equation is \( m^2 - k = 0 \Rightarrow m = \pm \sqrt{k} \)
Its solution is \( X(x) = c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x} \)

Similarly, solving (3), we get \( Y(y) = c_3 e^{(k-2)y} \)

Substituting these values in (1), we get

\[
\begin{align*}
\frac{\partial u}{\partial x} &= (c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}) c_3 e^{(k-2)y} \\
\frac{\partial^2 u}{\partial y^2} &= (A e^{\sqrt{k}x} + B e^{-\sqrt{k}x}) e^{(k-2)y} \quad \ldots (4)
\end{align*}
\]

where \( c_1 c_3 = A, \ c_2 c_3 = B \)

Given that \( u(0, y) = 0 \Rightarrow (A + B)e^{(k-2)y} = 0 \Rightarrow A + B = 0 \quad \text{or} \quad B = -A \)

\[
\therefore \ \frac{\partial u}{\partial x} = A\sqrt{k} \left( e^{\sqrt{k}x} + e^{-\sqrt{k}x} \right) e^{(k-2)y} \quad (\because B = -A)
\]

Also given that \( \frac{\partial u(0, y)}{\partial x} = 1 + e^{-3y} \Rightarrow A\sqrt{k} \left( e^{\sqrt{k}0} + e^{-\sqrt{k}0} \right) e^{(k-2)y} = 1 + e^{-3y} \)

\[
\Rightarrow 2A\sqrt{k} e^{(k-2)y} = e^{0y} + e^{-3y}
\]

This is possible only if either (i) \( k - 2 = 0 \) when \( 2A\sqrt{k} = 1 \) or (ii) \( k - 2 = -3 \) when \( 2A\sqrt{k} = 1 \)

(i) \( k - 2 = 0 \) when \( 2A\sqrt{k} = 1 \Rightarrow k = 2 \) and \( A = \frac{1}{2\sqrt{2}} \)

From (4), \( u(x, y) = \frac{1}{2\sqrt{2}} (e^{\sqrt{2}x} - e^{-\sqrt{2}x}) e^{0-3y} \) \( \text{i.e.,} \ u(x, y) = \frac{1}{\sqrt{2}} \sinh \sqrt{2}x \ \ldots (5) \)

(ii) \( k - 2 = -3 \) when \( 2A\sqrt{k} = 1 \Rightarrow k = -1 \) and \( A = \frac{1}{2\sqrt{-1}} = \frac{1}{2i} \) \( (\because i^2 = -1) \)

From (4), \( u(x, y) = \frac{1}{2i} (e^{ix} - e^{-ix}) e^{(-1-2)y} \) \( \text{i.e.,} \ u(x, y) = e^{-3y} \sin x \ \ldots (6) \)

By the principle of superposition, the required solution is \( u(x, y) = \frac{1}{\sqrt{2}} \sinh \sqrt{2}x + e^{-3y} \sin x \)

Exercise 10

Solve the following by the method of separation of variables:

1. \( \frac{\partial u}{\partial x} = 2 \frac{\partial u}{\partial t} + u \) where \( u(x, 0) = 6e^{-3x} \)
2. \( x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} = 0 \)
3. \( 3 \frac{\partial u}{\partial x} + 2 \frac{\partial u}{\partial y} = 0 \), where \( u(x, 0) = 4e^{-x} \)
4. \( 4u_x + u_y = 3u \) and \( u(0, y) = e^{-5y} \)
5. \( 4 \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u \) given \( u(0, y) = 3e^{-y} - e^{-5y} \)
6. \( u_{xt} = e^{-t} \cos x \) with \( u(x, 0) = 0 \) and \( \frac{\partial u(0, y)}{\partial t} = 0 \)

Answers:

1. \( u(x, t) = 6e^{-(3x+2t)} \)
2. \( u(x, y) = Ae^{k \left( \frac{1}{2} - \frac{1}{2} \right)} \)
3. \( u(x, y) = 4e^{-\frac{1}{2}(2x-3y)} \)
4. \( u(x, y) = e^{2x-5y} \)
5. \( u(x, y) = 3e^{x-y} - e^{2x-5y} \)
6. \( u(x, t) = \sin x(1-e^{-t}) \)
A4011 - PARTIAL DIFFERENTIAL EQUATIONS AND COMPLEX VARIABLES

Handout # 11

Limit, continuity, differentiability and analyticity of functions of a complex variable

Complex variable: If $x$ and $y$ are real variables, then $z = x + iy$ is called a complex variable. Here $x$ is called real part of $z$, denoted by $\text{Re}(z)$ and $y$ is called Imaginary part of $z$, denoted by $\text{Im}(z)$. If $z = x + iy$ then $\overline{z} = x - iy$ is called the complex conjugate of $z$.

Note: i) $\text{Re}(z) = \frac{z + \overline{z}}{2}$

ii) $\text{Im}(z) = \frac{z - \overline{z}}{2i}$

iii) $z \cdot \overline{z} = |z|^2 = x^2 + y^2$

Polar form of a complex variable: Every complex variable $z = x + iy$ can be represented in polar form as $z = re^{i\theta}$, where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\varepsilon$–Neighborhood of a point ($\varepsilon$–disc): The set $\{z : |z - z_0| < \varepsilon, \varepsilon > 0\}$ is called $\varepsilon$–neighborhood of a point $z_0$.

The set $\{z : 0 < |z - z_0| < \varepsilon, \varepsilon > 0\}$ is called deleted $\varepsilon$–neighborhood of a point $z_0$.

Note: $|z - z_0| = r$ represents a circle with centre at a point $z_0$ having radius $r$

Function of a complex variable: If corresponding to each value of a complex variable $z(= x + iy)$, there corresponds one or more values of $w(= u + iv)$, then $w$ is said to be a complex function of $z$ and we write as $w = f(z) = u(x, y) + iv(x, y)$.

If to each value of $z$, there corresponds one and only one value of $w$ then $w$ is said to be a single value function of $z$, otherwise, a multi valued function of $z$.

Example: (i) $w = \frac{1}{z}$ is a single valued function

(ii) $w = \log z$ is a multi valued function

Limit: A complex number $l$ is said to be limit of $f(z)$ as $z$ approaches $z_0$ and is denoted by $\lim_{z \to z_0} f(z) = l$.

Continuity: A function $f(z)$ is said to be continuous at a point $z = z_0$ if $f(z_0)$ exists, $\lim_{z \to z_0} f(z)$ exists and $\lim_{z \to z_0} f(z) = f(z_0)$.

A function $f(z)$ is said to be continuous in a region $R$ if it is continuous at every point of $R$.

Differentiability: A function $f(z)$ is said to be differentiable at a point $z = z_0$ if the
limit \( f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \) exists. The limit \( f'(z_0) \) is known as the derivative of \( f(z) \) at \( z_0 \). If \( f'(z_0) \) exists, the limits must be same whatever be the direction along which \( z = z_0 \).

**Note:** In real variable, \( x \to x_0 \) along the real line (x-axis) either from left or right. For a complex variable \( z \to z_0 \) along any path straight or curved since the points representing \( z \) and \( z_0 \) in the complex plane can be joined by an infinite no of ways.

**Analyticity:** A function \( f(z) \) is said to be analytic at a point \( z_0 \) if \( f(z) \) is differentiable not only at \( z_0 \) but at every point of some neighborhood of \( z_0 \).

A function \( f(z) \) is said to be analytic in a region \( R \) if it is analytic at every point of \( R \).

An analytic function is also known as *regular function* or *holomorphic function*.

**Entire function:** A function which is analytic everywhere in the complex plane is known as entire function.

**Examples:** Polynomials, \( \cos z, \sin z, e^z, etc. \)

**Note:** (i) If \( f(z) \) is analytic at \( z_0 \) then \( f(z) \) is differentiable and hence continuous at \( z_0 \)

(ii) If \( f(z) \) is differentiable at \( z_0 \) then \( f(z) \) need not be analytic at \( z_0 \)

\[ f(z) = |z|^2 \] is differentiable only at \( z = 0 \). So it is not analytic at \( z = 0 \).

(ii) If \( f(z) \) is not differentiable at \( z_0 \) then \( f(z) \) is not be analytic at \( z_0 \)

**Theorem:** If \( f(z) = u(x, y) + iv(x, y) \) is differentiable at any point \( z \) then at this point the first order partial derivatives \( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \) exist and satisfy the Cauchy-Riemann (C-R) equations

\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \]

**Note:** (i) C-R equations are used to determine whether a complex function is analytic or not *i.e.*, If \( f(z) = u + iv \) is analytic in a region \( R \) then \( u, v \) satisfy C-R equations at all points of \( R \).

(ii) C-R equations are necessary but not sufficient *i.e.*, If \( u, v \) satisfy C-R equations at a point \( z_0 \) then \( f(z) = u + iv \) need not be analytic at \( z_0 \).

(iii) C-R equations are sufficient if the partial derivatives are continuous *i.e.*, If the first order partial derivatives \( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \) are continuous and satisfy C-R equations at a point \( z_0 \) then \( f(z) = u + iv \) is analytic at \( z_0 \).

**C-R equations in polar form:** Cauchy-Riemann equations in polar form are given by
\( \frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \) and \( \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta} \)

**Example 11.1.** Show that \( \lim_{z \to 0} \frac{x^2 y}{x^2 + y^2} \) does not exist even though this function approaches the same limit along every straight line through the origin.

**Solution:**

**Path I:** Suppose \( z \to 0 \) along the straight line \( y = mx \)

\[
\lim_{z \to 0} \frac{x^2 y}{x^4 + y^2} = \lim_{x \to 0} \frac{x^2 y^2}{x^4 + y^2} = \lim_{x \to 0} \frac{x^2 (mx)^2}{x^4 + (mx)^2} = \lim_{x \to 0} \frac{mx}{x^2 + m^2} = \frac{m(0)}{0 + m^2} = 0
\]

**Path II:** Suppose \( z \to 0 \) along the parabola \( y = mx^2 \)

\[
\lim_{z \to 0} \frac{x^2 y}{x^4 + y^2} = \lim_{x \to 0} \frac{x^2 y^2}{x^4 + y^2} = \lim_{x \to 0} \frac{x^2 (mx^2)^2}{x^4 + (mx^2)^2} = \lim_{x \to 0} \frac{m}{1 + m^2} = \frac{m}{1 + m^2} \neq 0
\]

which depends on \( m \) and is not unique. Therefore the limit does not exist.

**Example 11.2.** Discuss the continuity of \( f(z) = \frac{z}{|z|} \) at the origin.

**Solution:** \( \lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{z}{|z|} \)

**Path I:** Suppose \( z \to 0 \) along the path in which \( y \to 0 \) first and then \( x \to 0 \).

\[
\lim_{z \to 0} \frac{z}{|z|} = \lim_{y \to 0} \frac{x + iy}{\sqrt{x^2 + y^2}} = \lim_{x \to 0} \frac{x + i(0)}{\sqrt{x^2 + 0}} = \lim_{x \to 0} \frac{1}{\sqrt{1}} = 1 \neq 0
\]

**Path II:** Suppose \( z \to 0 \) along the path in which \( x \to 0 \) first and then \( y \to 0 \).

\[
\lim_{z \to 0} \frac{z}{|z|} = \lim_{x \to 0} \frac{x + iy}{\sqrt{x^2 + y^2}} = \lim_{y \to 0} \frac{0 + i(0)}{\sqrt{0 + y^2}} = \lim_{y \to 0} (i) = i \neq 0
\]

Since limit does not exist at the origin, \( f(z) \) is not continuous at the origin.

**Example 11.3.** Show that \( f(z) = z + 2\bar{z} \) is not analytic anywhere in the complex plane.

**Solution:** Let \( f(z) = z + 2\bar{z} = x + iy + 2(x - iy) = 3x - iy = u + iv \),

where \( u = 3x \) and \( v = -y \)

\[
\therefore \frac{\partial u}{\partial x} = 3, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0 \text{ and } \frac{\partial v}{\partial y} = -1
\]

Clearly, \( \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \) and \( \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \) i.e., C-R equations are not satisfied

\( \therefore f(z) \) is not analytic anywhere in the complex plane.

**Example 11.4.** Find the constants \( l \) and \( m \) so that the function \( f(z) \) should be analytic, where

\[
f(z) = (x^2 + ly^2 - 2xy) + i(mx^2 - y^2 + 2xy)
\]
Solution: Given \( f(z) = (x^2 + iy^2 - 2xy) + i(mx^2 - y^2 + 2xy) = u + iv, \)
where \( u = x^2 + iy^2 - 2xy, \quad v = mx^2 - y^2 + 2xy \)
\[ \frac{\partial u}{\partial x} = 2x - 2y, \quad \frac{\partial u}{\partial y} = 2ly - 2x, \quad \frac{\partial v}{\partial x} = 2xm + 2y, \quad \frac{\partial v}{\partial y} = -2y + 2x \]

Since \( f(z) = u + iv \) is analytic, \( u \) and \( v \) must satisfy C-R equations \( \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \) and \( \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}. \)

Now \( \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow 2ly - 2x - 2xm - 2y \Rightarrow l = -1 \) and \( m = 1 \)

Example 11.5. Show that for \( f(z) = \begin{cases} 2xy(x+iy) & \text{if } z \neq 0 \\ x^2+y^2 & \text{if } z = 0 \end{cases} \)
C-R equations are satisfied at the origin but derivative of \( f(z) \) at origin does not exist.

Solution: Given \( f(z) = \frac{2xy(x+iy)}{x^2+y^2} = u(x, y) + iv(x, y), \)
where \( u(x, y) = \frac{2x^2 y}{x^2+y^2}, \quad v(x, y) = \frac{2xy^2}{x^2+y^2}. \) Also \( f(0) = 0 \Rightarrow u(0,0) = 0 \) and \( v(0,0) = 0 \)

\[ f_z = \lim_{z \to 0} \frac{f(z) - f(0)}{z-0} = \lim_{z \to 0} \frac{2xy}{x^2+y^2} = 0 \]

Path I: Suppose \( z \to 0 \) along the path in which \( y \to 0 \) first and then \( x \to 0. \)
\[ f''(0) = \lim_{z \to 0} \frac{2xy}{x^2+y^2} = \lim_{x \to 0} \frac{2x}{x^2+y^2} = \lim_{x \to 0} 2x(0) = \lim_{x \to 0} (0) = 0 \]

Path II: Suppose \( z \to 0 \) along the path in which \( x \to 0 \) first and then \( y \to 0. \)
$$f'(0) = \lim_{z \to 0} \frac{2xy}{x^2+y^2} = \lim_{y \to 0} \frac{2xy}{x^2+y^2} = \lim_{y \to 0} \frac{2(0,y)}{y^2} = \lim_{y \to 0}(0) = 0$$

Path III: Suppose $z \to 0$ along the straight line $y = mx$

$$f'(0) = \lim_{y=m(x)} \frac{2xy}{x^2+y^2} = \lim_{x \to 0} \frac{2(mx)2(mx)^2}{x^2+(mx)^2} = \lim_{x \to 0} \frac{2m}{1+m^2} = \frac{2m}{1+m^2} \neq 0$$

Thus derivative of $f(z)$ does not exist at $z=0$.

**Example 11.6.** Show that $z^n (n \in Z^+)$ is analytic. Hence find its derivative.

**Solution:** Let $f(z) = z^n = (r e^{i\theta})^n = r^n e^{i\theta} = r^n (\cos n\theta + i \sin n\theta) = u(r, \theta) + iv(r, \theta)$, ... (1)

where $u = r^n \cos n\theta$, $v = r^n \sin n\theta$

$\therefore \frac{\partial u}{\partial r} = nr^{n-1} \cos n\theta$, $\frac{\partial u}{\partial \theta} = -nr^n \sin n\theta$, $\frac{\partial v}{\partial r} = nr^{n-1} \sin n\theta$ and $\frac{\partial v}{\partial \theta} = nr^{n-1} \cos n\theta$

We have $\frac{\partial u}{\partial r} = nr^{n-1} \cos n\theta = \frac{1}{r} \{nr^n \cos n\theta\} = \frac{1}{r} \frac{\partial v}{\partial \theta}$

and $\frac{\partial v}{\partial r} = nr^{n-1} \sin n\theta = -\frac{1}{r} \{-nr^n \sin n\theta\} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$

Clearly, C-R equations are satisfied. Moreover, the partial derivatives $\frac{\partial u}{\partial r}$, $\frac{\partial u}{\partial \theta}$, $\frac{\partial v}{\partial r}$, $\frac{\partial v}{\partial \theta}$ are continuous.

Hence $z^n$ is analytic for $n \in Z^+$.

**Derivative of $f(z)$:**

Differentiating (1) partially w. r. t. $r$, we get

$$f'(z) \cdot \frac{\partial r}{\partial \theta} = \frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r}$$

i.e., $f'(z) \cdot e^{i\theta} = nr^{n-1} \cos n\theta + i(nr^{n-1} \sin n\theta)$

$\therefore z = re^{i\theta} \Rightarrow \frac{\partial z}{\partial r} = e^{i\theta}$

$\Rightarrow f'(z) = \frac{nr^{n-1}}{e^{i\theta}} \cos n\theta + i \sin n\theta = \frac{nr^{n-1}}{e^{i\theta}} e^{i\theta} = nr^{n-1}e^{i(n-1)\theta} = n \left(re^{i\theta}\right)^{n-1} = nz^{n-1}$

$\therefore f'(z) = nz^{n-1} (n \in Z^+)$

**Exercise 11**

1. Find $a,b,c,d$ and $e$ if

$$f(z) = \left(ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2\right) + i\left(4x^3y - exy^3 + 4xy\right)$$

2. Determine $p$ such that the function $f(z) = \frac{1}{2} \log(x^2 + y^2) + i \tan^{-1}\left(\frac{px}{y}\right)$ be analytic function.

3. Find all the values of $k$ such that $f(z) = e^x (\cos ky + i \sin ky)$ is analytic.

4. Show that $e^z$ is analytic everywhere in the complex plane and find its derivative.
5. If \( w = \log z \), find \( \frac{dw}{dz} \) and determine where \( w \) is non-analytic.

6. Show that the function \( f(z) = \sqrt{|xy|} \) is not analytic at the origin even though C-R equations are satisfied thereof.

7. Prove that the function \( f(z) \) defined by
   \[
   f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} \quad (z \neq 0), \quad f(0) = 0
   \]
   is continuous and the C-R equations are satisfied at the origin, yet \( f'(0) \) does not exist.

8. Show that the function
   \[
   f(z) = \frac{xy^2(x + iy)}{x^2 + y^4}, \quad z \neq 0 \quad \text{and} \quad f(0) = 0
   \]
   is not analytic at \( z = 0 \) although C-R equations are satisfied at \( z = 0 \).

9. If \( f(z) = \begin{cases} \frac{x^3(y - ix)}{x^2 + y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases} \), prove that
   \[
   \frac{f(z) - f(0)}{z} \to 0 \quad \text{as} \quad z \to 0
   \]
   along any radius vector but not as \( z \to 0 \) along the curve \( y = ax^3 \).

10. Show that the function
    \[
    f(z) = \begin{cases} \frac{(z)^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}
    \]
    satisfied at \((0,0)\) but not differentiable at \((0,0)\).

Answers:

1. \( a = 1, \quad b = -6, \quad c = 1, \quad d = 2, \quad e = 4 \)
2. \( p = -1 \)
3. \( k = 1 \)
4. \( w \) is not analytic at \( z = 0 \) and \( \frac{dw}{dz} = \frac{1}{z} (z \neq 0) \)
Milne-Thompson method & Harmonic and conjugate harmonic functions

Milne-Thompson method: Let \( f(z) = u(x, y) + iv(x, y) \) \ldots (1)

Since \( z = x + iy \) and \( \bar{z} = x - iy \), we have
\[
x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i},
\]
Equation (1) becomes
\[
f(z) = u \left( \frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i} \right) + iv \left( \frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i} \right)
\]
Consider this as an identity in two independent variables \( z, \bar{z} \) and putting \( z = \bar{z} \), we get
\[
f(z) = u(z, 0) + iv(z, 0) \ldots (2)
\]
It is clear that (2) is same as (1), if we replace \( x \) by \( z \) and \( y \) by 0.
Thus to express any function in terms of \( z, \bar{z} \), replace \( x \) by \( z \) and \( y \) by 0. This provides an elegant method of finding \( f(z) \) when its real part or the imaginary part is given.

Construction of analytic function whose real or imaginary part is known:

Suppose \( f(z) = u(x, y) + iv(x, y) \) \ldots (1) be analytic function whose real part \( u(x, y) \) is known

1. Find \( u_x = \phi_1(x, y) \) and \( u_y = \phi_2(x, y) \)
2. Differentiating (1) partially w. r. t. \( x \), we get
\[
f'(z) = u_x + iv_x
\]
\[
= u_x - iu_y \quad [\because \text{By C-R equations, } u_y = -v_x]
\]
\[
\therefore f'(z) = \phi_1(x, y) + i\phi_2(x, y) \ldots (2)
\]
3. By Milne-Thompson method \( f'(z) \) can be expressed in terms of \( z \) by replacing \( x \) with \( z \) and \( y \) with 0.
Then (2) becomes
\[
f'(z) = \phi_1(z, 0) + i\phi_2(z, 0) \ldots (3)
\]
4. Integrating (3), we get the required analytic function
\[
f(z) = \int [\phi_1(z, 0) + i\phi_2(z, 0)] \, dz + c
\]

Exercise 12.1. Determine the analytic function whose real part is \( e^{2x}(x \cos 2y - y \sin 2y) \)

Solution. Suppose \( f(z) = u(x, y) + iv(x, y) \) \ldots (1) be analytic function, where
\[
u(x, y) = e^{2x}(x \cos 2y - y \sin 2y)
\]
\[
\therefore u_x = e^{2x}[(2x + 1) \cos 2y - 2y \sin 2y] \quad \text{and} \quad u_y = -e^{2x}[(2x + 1) \sin 2y + y \cos 2y]
\]
Differentiating (1) w. r. t. \( x \), we get
\[
f'(z) = u_x + iv_x
\]
\[
= u_x - iv_y \quad [\because \text{By C-R equations, } u_y = -v_x]
\]
\[
\therefore f'(z) = e^{2z}[(2x + 1) \cos 2y - 2y \sin 2y] - i e^{2z}[(2x + 1) \sin 2y + y \cos 2y] \ldots (2)
\]
By Milne-Thompson method \( f'(z) \) can be expressed in terms of \( z \) by replacing \( x \) with \( z \) and \( y \) with 0.
Then (2) becomes
\[
f'(z) = e^{2z}(2z + 1)
\]
Integrating,
\[
f(z) = \int e^{2z}(2z + 1) \, dz + ic
\]
\[
= \frac{e^{2z}}{2}[(2z + 1) - 1] + ic = e^{2z}z + ic
\]

Exercise 12.2. Find the analytic function \( f(z) = u + iv \), if \( u - v = \frac{\cos x + \sin x - e^{-y}}{2 \cos x - e^{-y} - e^y} \) and \( f\left(\frac{\pi}{2}\right) = 0 \).

**Solution.** Suppose \( f(z) = u(x, y) + iv(x, y) \ldots (1) \) be analytic function,
where \( u - v = \frac{\cos x + \sin x - e^{-y}}{2 \cos x - e^{-y} - e^y} = \frac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)} \)
\[
\therefore u_x - v_x = \frac{(\cos x - \cosh y)(-\sin x + \cos x) - (\cos x + \sin x - e^{-y})(-\sin x)}{2(\cos x - \cosh y)^2}
\]
\[
= \frac{(\sin x - \cos x)\cosh y - e^{-y}\sin x + 1}{2(\cos x - \cosh y)^2} \ldots (2)
\]
and \( u_y - v_y = \frac{(\cos x - \cosh y)e^{-y} - (\cos x + \sin x - e^{-y})(-\sinh y)}{2(\cos x - \cosh y)^2} \)
or \( -v_x - u_x = \frac{(\cos x - \cosh y)e^{-y}+(\cos x + \sin x - e^{-y})\sinh y}{2(\cos x - \cosh y)^2} \ldots (3) \)
Subtracting (3) from (2), we get
\[
u_x = \frac{(\sin x - \cos x)\cosh y - e^{-y}\sin x + 1-(\cos x - \cosh y)e^{-y}-(\cos x + \sin x - e^{-y})\sinh y}{4(\cos x - \cosh y)^2}
\]
Adding (2) and (3), we get
\[
v_x = \frac{(\sin x - \cos x)\cosh y - e^{-y}\sin x + 1+(\cos x - \cosh y)e^{-y}+(\cos x + \sin x - e^{-y})\sinh y}{4(\cos x - \cosh y)^2}
\]
Differentiating (1) w. r. t. \( x \), we get
\[
f'(z) = u_x + iv_x
\]
\[
f'(z) = \frac{(\sin x - \cos x)\cosh y - e^{-y}\sin x + 1-(\cos x - \cosh y)e^{-y}-(\cos x + \sin x - e^{-y})\sinh y}{4(\cos x - \cosh y)^2}
\]
\[
+ i \frac{(\sin x - \cos x)\cosh y - e^{-y}\sin x + 1+(\cos x - \cosh y)e^{-y}+(\cos x + \sin x - e^{-y})\sinh y}{4(\cos x - \cosh y)^2} \ldots (4)
\]
By Milne-Thompson method \( f'(z) \) can be expressed in terms of \( z \) by replacing \( x \) with \( z \) and \( y \) with 0. Then (4) becomes

\[
f'(z) = \frac{(\sin z - \cos z) \cosh 0 - e^{-0} \sin z + 1 - (\cos z - \cosh 0) e^{-0} - (\cos z + \sin z - e^{-0}) \sinh 0}{4(\cos z - \cosh 0)^2}
\]
\[+ i \frac{(\sin z - \cos z) \cosh 0 - e^{-0} \sin z + 1 + (\cos z - \cosh 0) e^{-0} + (\cos z + \sin z - e^{-0}) \sinh 0}{4(\cos z - \cosh 0)^2}
\]

\[
f'(z) = \frac{2(1-\cos z)}{4(\cos z - 1)^2} = \frac{(1-\cos z)^2}{2(1-\cos z)} = \frac{1}{2(1-\cos z)} = \frac{1}{4 \sin^2 \frac{z}{2}} = \frac{1}{4} \csc^2 \frac{z}{2}
\]

Integrating, \( f(z) = \frac{1}{2} \int \frac{1}{\csc \frac{z}{2}} \, dz + c = -\frac{1}{2} \cot \frac{z}{2} + c \ldots (5) \)

\[
f\left(\frac{\pi}{2}\right) = 0 \Rightarrow -\frac{1}{2} \cot \frac{\pi}{4} + c = 0 \quad \text{or} \quad c = \frac{1}{2}
\]

Hence \( f(z) = \frac{1}{2} \left(1 - \cot \frac{z}{2}\right) \)

**Harmonic function:** A function \( \phi(x, y) \) is said to be a harmonic if it satisfies the Laplace’s equation \( \text{i.e.,} \)

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.
\]

**Example:** \( \phi(x, y) = x^2 - y^2 \) is harmonic function

**Note:** A function \( u(r, \theta) \) is said to be harmonic if it satisfies the Laplace’s equation in polar coordinates

\[
i.e., \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0
\]

**Result 1.** The real and imaginary parts of an analytic function are harmonic.

**Proof.** Suppose \( f(z) = u + iv \) is an analytic function, then \( u \) and \( v \) must satisfy C-R equations

\[
i.e., \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ldots (1) \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \ldots (2)
\]

Differentiating (1) w. r. t. \( x \) and (2) w. r. t. \( y \), we obtain

\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} \ldots (3) \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y^2} \ldots (4)
\]

Adding (3) and (4) and assuming that \( \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} \), we get

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ldots (5)
\]

Similarly, by differentiating (1) w. r. t. \( y \) and (2) w. r. t. \( x \) and subtracting, we obtain

\[
\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \ldots (6)
\]

Since both \( u \) and \( v \) satisfy the Laplace’s equation, they are harmonic functions.
**Conjugate harmonic function:** The imaginary part \( v \) of an analytic function \( f(z) = u + iv \) is called the *harmonic conjugate* of the real part \( u \) and vice versa (i.e., \( u \) is harmonic conjugate of \( v \)).

**Exercise 12.3.** Show that \( u = \frac{1}{2} \log(x^2 + y^2) \) is harmonic and find its harmonic conjugate.

**Solution.** Given \( u(x, y) = \frac{1}{2} \log(x^2 + y^2) \)

\[
\therefore u_x = \frac{x}{x^2 + y^2}, u_{xx} = \frac{x^2 - y^2}{(x^2 + y^2)^2}, u_y = \frac{y}{x^2 + y^2}, u_{yy} = \frac{y^2 - x^2}{(x^2 + y^2)^2}
\]

Now \( u_{xx} + u_{yy} = \frac{x^2 - y^2}{(x^2 + y^2)^2} + \frac{y^2 - x^2}{(x^2 + y^2)^2} = 0 \)

Since \( u(x, y) \) satisfies Laplace’s equation, it is harmonic.

Let \( v(x, y) \) be the harmonic conjugate of \( u(x, y) \). Then \( u(x, y) \) and \( v(x, y) \) satisfy C-R equations i.e, \( u_x = v_y \) and \( u_y = -v_x \)

We know that \( dv = v_x dx + v_y dy \)

\[
= -u_y dx + u_x dy \quad [\because \text{By C-R equations}]
\]

\[
= -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy
\]

\[
= \frac{xdy - ydx}{x^2 + y^2}
\]

\[
= d\left[ \tan^{-1}\left( \frac{x}{y} \right) \right]
\]

Integrating, we get

\[
v(x, y) = \tan^{-1}\left( \frac{x}{y} \right) + c
\]

**Exercise 12.4.** Find the conjugate harmonic of \( v(r, \theta) = r^2 \cos 2\theta - r \cos \theta + 2 \). Show that \( v \) is harmonic.

**Solution.** Given

\[
v(r, \theta) = r^2 \cos 2\theta - r \cos \theta + 2. \therefore v_r = 2r \cos 2\theta - \cos \theta, v_{rr} = 2 \cos 2\theta, v_\theta = -2r^2 \sin 2\theta + r \sin \theta
\]

and \( v_{r\theta} = -4r^2 \cos 2\theta + r \cos \theta \).

Now \( v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{r\theta} = 2 \cos 2\theta + \frac{1}{r} (2r \cos 2\theta - \cos \theta) + \frac{1}{r^2} (-4r^2 \cos 2\theta + r \cos \theta) \)

\[
= 2 \cos 2\theta + 2 \cos 2\theta - \frac{1}{r} \cos \theta - 4 \cos 2\theta + \frac{1}{r} \cos \theta = 0
\]

Since \( v(r, \theta) \) satisfies Laplace’s equation, it is harmonic.

Let \( u(r, \theta) \) be the harmonic conjugate of \( v(r, \theta) \). Then \( u(r, \theta) \) and \( v(r, \theta) \) satisfy C-R equations i.e,

\( u_r = \frac{1}{r} v_\theta \) and \( v_r = -\frac{1}{r} u_\theta \)

We know that \( du = u_r dr + u_\theta d\theta \)
\[ v_0 dr - rv_r d\theta \quad \text{[\because \text{By C\textendash}R equations]} \]
\[ = \frac{1}{r} (-2r^2 \sin 2\theta + r \sin \theta) dr - r(2r \cos 2\theta - \cos \theta) d\theta \]
\[ = (-2r \sin 2\theta + \sin \theta) dr -(2r^2 \cos 2\theta - r \cos \theta) d\theta \]
\[ = -2(r \sin 2\theta \, dr + 2r^2 \cos 2\theta d\theta) + (\sin \theta dr + r \cos \theta d\theta) \]
\[ = -d(r^2 \sin 2\theta) + d(r \sin \theta) \]

Integrating, we get
\[ u(r, \theta) = -r^2 \sin 2\theta + r \sin \theta + c \]

**Note:** If \( \phi(x, y) \) is the velocity potential and \( \psi(x, y) \) is the stream function of an electric field then \( w = \phi + i\psi \) represents the complex potential function and is always analytic.

**Exercise 12.5.** If \( w = \phi + i\psi \) represents the complex potential for an electric field and \( \psi = x^2 - y^2 + \frac{x}{x^2 + y^2} \), determine the function \( \phi \).

**Solution.** Given \( \psi = x^2 - y^2 + \frac{x}{x^2 + y^2} \)

\[ \therefore \quad \psi_x = 2x + \frac{y^2 - x^2}{(x^2 + y^2)^2} \quad \text{and} \quad \psi_y = -2y - \frac{2xy}{(x^2 + y^2)^2} \]

Since \( w = \phi + i\psi \) represents the complex potential function, \( \phi(x, y) \) and \( \psi(x, y) \) satisfy Cauchy-Riemann equations \( i.e. \), \( \phi_x = \psi_y \) and \( \phi_y = -\psi_x \)

We know that \( d\phi = \phi_x \, dx + \phi_y \, dy \)

\[ = \psi_y \, dx - \psi_x \, dy \quad \text{[\because \text{By C\textendash}R equations]} \]

\[ = \left[ -2y - \frac{2xy}{(x^2 + y^2)^2} \right] dx - \left[ 2x + \frac{y^2 - x^2}{(x^2 + y^2)^2} \right] dy \]

\[ = -2(\psi dx + x \psi dy) + \frac{(x^2 - y^2) \, dy - 2xy \, dx}{(x^2 + y^2)^2} \]

\[ = -2d(xy) + \frac{(x^2 + y^2) \, dy - 2xy \, dx - 2y^2 \, dy}{(x^2 + y^2)^2} \]

\[ = -2d(xy) + \frac{(x^2 + y^2) \, dy - y(2dx + 2y \, dy)}{(x^2 + y^2)^2} \]

\[ = -2d(xy) + \frac{(x^2 + y^2) \, dy - yd(x^2 + y^2)}{(x^2 + y^2)^2} \]

\[ = -2d(xy) + d\left( \frac{y}{x^2 + y^2} \right) \]

Integrating, we get

\[ \phi(x, y) = -2xy + \frac{y}{x^2 + y^2} + c \]
Exercise 12

1. Determine the analytic function whose real part is

\( (i) \frac{y}{x^2 + y^2} \quad (ii) y + e^x \cos y \quad (iii) x \sin x \cosh y - y \cos x \sinh y \)

\( (iv) \frac{\sin 2x}{\cosh 2y - \cos 2x} \quad (v) x^3 - 3xy^2 + 3x^2 - 3y^2 \quad (vi) e^x \left[ (x^2 - y^2) \cos y - 2xy \sin y \right] \)

2. Find the regular function whose imaginary part is

\( (i) \frac{x-y}{x^2 + y^2} \quad (ii) -\sin x \sin y \quad (iii) e^{-x}(x \cos y + y \sin y) \quad (iv) \frac{2 \sin x \sin y}{\cos 2x + \cosh 2y} \)

3. Find the analytic function \( f(z) = u + iv \), if

\( (i) u - v = (x - y)(x^2 + 4xy + y^2) \)

\( (ii) u - v = e^x (\cos y - \sin y) \)

\( (iii) u + v = \frac{2 \sin 2x}{e^{2y} + e^{-2y} - 2 \cos 2x} \)

4. Find an analytic function \( f(z) = u + iv \) if \( v(r, \theta) = \left( r - \frac{1}{r} \right) \sin \theta, r \neq 0 \)

5. Show that the function \( u = e^x \cos y \) is harmonic and find its harmonic conjugate.

6. Show that the function \( u(r, \theta) = e^{-\theta} \cos (\ln r) \) is harmonic and find its harmonic conjugate.

7. Show that the function \( f(x, y) = x^3 y - xy^3 + xy + x + y \) can be the imaginary part of an analytic function. Determine the real part.

8. Show that the function \( u = e^{-2xy} \sin(x^2 - y^2) \) is harmonic. Find the harmonic conjugate \( \psi \) and express \( u + iv \) as an analytic function of \( z \).

9. In a two dimensional fluid flow, the velocity potential is \( \phi = 3x^2 y - y^3 \), find the stream function \( \psi \).

10. Prove that \( u = x^2 - y^2 \) and \( v = \frac{y}{x^2 + y^2} \) are harmonic functions but are not harmonic conjugates.

**Answers:**

1. \( (i) \frac{i}{z} + ic \quad (ii) e^z - iz + ic \quad (iii) z \sin z + ic \quad (iv) \cot z + ic \)

\( (v) z^3 + 3z^2 + 1 + ic \quad (vi) z^2 e^z + ic \)

2. \( (i) \frac{1+i}{z} + c \quad (ii) \cos z + c \quad (iii) 1 + iz e^{-z} + c \quad (iv) \sec z + c \)

3. \( (i) -iz^3 + c \quad (ii) e^z + c \quad (iii) \frac{1+i}{z} \cot z + c \)

4. \( f(z) = \left( r + \frac{1}{r} \right) \cos \theta + i \left( r - \frac{1}{r} \right) \sin \theta + c \)

5. \( v = e^x \sin y + c \)
6. \( v(r, \theta) = e^{-\theta} \sin(\ln r) + c \)

7. \( g(x, y) = \frac{1}{4}(+ x^4 + y^4 - 6x^2y^2 + 2x^2 - 2y^2 + 6x - 6y) + c \)

8. \( v = -e^{-2xy} \cos(x^2 - y^2) + c; \quad f(z) = -ie^{iz^2} + ic \)

9. \( \psi = x^3 - 3xy^2 + c \)
Some important results on analytic functions

Orthogonal System: Two families of curves said to form an orthogonal system if they intersect at right angle at each point of their intersection.

**Result 1.** Every analytic function \( f(z) = u + iv \) defines two families of curves \( u(x, y) = c_1 \) and \( v(x, y) = c_2 \), which form an orthogonal system, where \( c_1 \) and \( c_2 \) are constants.

**Proof.** Consider the two families of curves \( u(x, y) = c_1 \) . . . (1) and \( v(x, y) = c_2 \) . . . (2)

Differentiating (1) w. r. t. \( x \), we get

\[
\frac{du}{dx} + \frac{du}{dy} \cdot \frac{dy}{dx} = 0
\]

or

\[
\frac{dy}{dx} = -\frac{\frac{du}{dx}}{\frac{du}{dy}} = m_1 \text{ (say) } [\because \text{ By C-R equations}]
\]

Similarly (2) gives

\[
\frac{dy}{dx} = -\frac{\frac{dv}{dx}}{\frac{dv}{dy}} = m_2 \text{ (say)}
\]

\( \therefore m_1 m_2 = -1 \) i.e., (1) and (2) cut orthogonally.

Hence two families of curves \( u(x, y) = c_1 \) and \( v(x, y) = c_2 \) form an orthogonal system.

**Example 13.1.** For \( w = \exp(z^2) \), find \( u \) and \( v \), and prove that the curves \( u(x, y) = c_1 \) and \( v(x, y) = c_2 \) where \( c_1 \) and \( c_2 \) are constants, cut orthogonally.

**Solution.** Given \( w = e^{z^2} = e^{(x+iy)^2} = e^{(x^2-y^2+i2xy)} = e^{x^2-y^2} \cdot e^{i2xy} = e^{x^2-y^2} (\cos 2xy + i \sin 2xy) = u(x, y) + iv(x, y) \),

where \( u(x, y) = e^{x^2-y^2} \cos 2xy \) and \( v(x, y) = e^{x^2-y^2} \sin 2xy \).

\( u(x, y) = c_1 \Rightarrow e^{x^2-y^2} \cos 2xy = c_1 \) . . . (1) and \( v(x, y) = c_2 \Rightarrow e^{x^2-y^2} \sin 2xy = c_2 \) . . . (2)

Differentiating (1) w. r. t. \( x \), we get

\[
2e^{x^2-y^2} (x \cos 2xy - y \sin 2xy) - 2e^{x^2-y^2} (y \cos 2xy + x \sin 2xy) \cdot \frac{dy}{dx} = 0 \quad [\because \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0]
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{x \cos 2xy - y \sin 2xy}{y \cos 2xy + x \sin 2xy} = m_1 \text{ (say)}
\]

Differentiating (2) w. r. t. \( x \), we can get

\[
\frac{dy}{dx} = \frac{y \cos 2xy + x \sin 2xy}{y \sin 2xy - x \cos 2xy} = m_2 \text{ (say)}
\]

\( \therefore m_1 m_2 = \frac{(x \cos 2xy - y \sin 2xy)}{(y \cos 2xy + x \sin 2xy)} \cdot \frac{(y \cos 2xy + x \sin 2xy)}{(y \sin 2xy - x \cos 2xy)} = -1 \)
Hence two families of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ cut orthogonally.

**Example 13.2.** Show that the curves $r^n = \alpha \sec n\theta$ and $r^n = \beta \csc n\theta$ cut orthogonally, where $\alpha$ and $\beta$ are constants.

**Solution.** Given curves can be written as $r^n \cos n\theta = \alpha$ and $r^n \sin n\theta = \beta$

Write $u(r, \theta) = r^n \cos n\theta$ and $v(r, \theta) = r^n \sin n\theta$

We have $f(z) = u(r, \theta) + iv(r, \theta) = r^n \cos n\theta + ir^n \sin n\theta$

$$= r^n (\cos n\theta + i\sin n\theta) = r^n e^{in\theta} = (re^{i\theta})^n = z^n$$

Since $f(z) = z^n$ is analytic, the curves $u(r, \theta) = \alpha$ and $v(r, \theta) = \beta$ cut orthogonally.

**Example 13.3.** Find the orthogonal trajectories of the family of curves $e^x \cos y - xy = c$

Take $u(x, y) = e^x \cos y - xy \ldots$ (1) then $v(x, y) =$ constant will be the required family of orthogonal trajectories if $f(z) = u(x, y) + iv(x, y)$ is analytic.

Differentiating (1) w. r. t. $x$ and $y$, we get

$$u_x = e^x \cos y - y \quad \text{and} \quad u_y = -e^x \sin y - x$$

Since $f(z)$ is analytic, $u$ and $v$ satisfy C-R equations i.e., $u_x = v_y$ and $u_y = -v_x$

We know that $dv = v_x dx + v_y dy$

$$= -u_y dx + u_x dy \quad \left[ \because \text{By C-R equations} \right]$$

$$= (e^x \sin y + x)dx + (e^x \cos y - y)dy$$

$$= (e^x \sin y dx + e^x \cos y dy) + xdx - ydy$$

$$= d(e^x \sin y) + xdx - ydy$$

Integrating, we get

$$v(x, y) = e^x \sin y - \frac{y^2}{2} + \frac{x^2}{2} + c_1$$

Hence the required family of orthogonal trajectories is $x^2 - y^2 + 2e^x \sin y = d$

**Example 13.4.** Find the orthogonal trajectories of the family of curves $r^2 \cos 2\theta = c$

**Solution.** Take $u(r, \theta) = r^2 \cos 2\theta \ldots$ (1) then $v(r, \theta) =$ constant will be the required family of orthogonal trajectories if $f(z) = u(r, \theta) + iv(r, \theta)$ is analytic.

Differentiating (1) w. r. t. $r$ and $\theta$, we get

$$u_r = 2r \cos 2\theta \quad \text{and} \quad u_\theta = -2r^2 \sin 2\theta$$

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Since \( f(z) \) is analytic, \( u \) and \( v \) satisfy C-R equations i.e., \( u_r = \frac{1}{r} v_\theta \) and \( v_r = -\frac{1}{r} u_\theta \)

We know that \( dv = v_r dr + v_\theta d\theta \)

\[
= -\frac{1}{r} u_\theta dr + ru_r d\theta \quad [\because \text{By C-R equations}]
\]

\[
= -\frac{1}{r} (-2r^2 \sin 2\theta) dr + r(2r \cos 2\theta) d\theta
\]

\[
= 2r \sin 2\theta dr + 2r^2 \cos 2\theta d\theta
\]

\[
= d(r^2 \sin 2\theta)
\]

Integrating, we get \( v(r, \theta) = r^2 \sin 2\theta + c_1 \)

Hence the required family of orthogonal trajectories is \( r^2 \sin 2\theta = d \).

**Example 13.5.** If \( f(z) \) is an analytic function with constant modulus, show that \( f(z) \) is constant.

or

Show that an analytic function with constant absolute value is constant.

**Solution.** Suppose \( f(z) = u + iv \) is an analytic function, then

\[
|f(z)| = \sqrt{u^2 + v^2} = k(\text{constant}) \Rightarrow u^2 + v^2 = k^2 \quad \ldots (1)
\]

Differentiating (1) partially w. r. t. \( x \) and \( y \), we get

\[
2uu_x + 2vv_x = 0 \quad \text{and} \quad 2uu_y + 2vv_y = 0
\]

or

\[
uu_x + vv_x = 0 \quad \ldots (2) \quad \text{and} \quad uu_y + vv_y = 0 \quad \ldots (3)
\]

Using C-R equations \( uu_x = vv_y \) and \( uu_y = -vv_x \), (3) becomes

\[
-uv_x + vu_x = 0 \quad \ldots (4)
\]

Squaring and adding (2) and (4), we obtain

\[
u^2(u_x)^2 + v^2(v_x)^2 + u^2(v_x)^2 + v^2(u_x)^2 = 0
\]

or

\[
(u^2 + v^2)[(u_x)^2 + (v_x)^2] = 0 \quad \ldots (5)
\]

**Case (i):** Suppose \( k = 0 \) i.e., \( u^2 + v^2 = 0 \Rightarrow u = 0 \) and \( v = 0 \)

\[
\therefore f(z) = u + iv = 0 \text{, which is constant.}
\]

**Case (ii):** Suppose \( k \neq 0 \) i.e., \( u^2 + v^2 \neq 0 \) then (5) implies that \( (u_x)^2 + (v_x)^2 = 0 \quad \ldots (6) \)

But \( f'(z) = u_x + iv_x \Rightarrow |f'(z)|^2 = (u_x)^2 + (v_x)^2 \)

\[
\Rightarrow |f'(z)|^2 = 0
\]

\[
\Rightarrow f(z) = \text{constant}
\]
Example 13.6. If $f(z)$ is analytic function of $z$, prove that $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|\text{Re} f(z)|^2 = 2|f'(z)|^2$$

Solution. Suppose $f(z) = u + iv$ is an analytic function, then $|\text{Re} f(z)| = u$

We have $\frac{\partial}{\partial x}(u^2) = 2uu_x$ and $\frac{\partial^2}{\partial x^2}(u^2) = 2\left[(u_x)^2 + uu_{xx}\right] \ldots (2)$

Similarly, we can get $\frac{\partial^2}{\partial y^2}(u^2) = 2\left[(u_y)^2 + uu_{yy}\right] \ldots (3)$

(2) + (3) gives $$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}u^2 = 2\left[(u_x)^2 + (u_y)^2 + u(u_{xx} + u_{yy})\right] \ldots (4)$$

Since $f(z)$ is analytic, $u$ satisfies the Laplace’s equation $u_{xx} + u_{yy} = 0 \ldots (5)$

From (4) and (5), we have $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|\text{Re} f(z)|^2 = 2|f'(z)|^2 \ldots (6)$$

Example 13.7. If $f(z)$ is analytic function of $z$, prove that $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2 \left|f(z)\right|^{p-2} \left|f'(z)\right|^2$$

Solution. Suppose $f(z) = u + iv$ is an analytic function, then $|f(z)|^p = (u^2 + v^2)^{p/2} = \phi(x, y)$, say

$$\therefore \frac{\partial \phi}{\partial x} = p(u^2 + v^2)^{p/2-1}(uu_x + vv_x) \quad \text{and}$$

$$\frac{\partial^2 \phi}{\partial x^2} = p\left(\frac{p}{2} - 1\right)(u^2 + v^2)^{p/2-2} \cdot 2(uu_x + vv_x)^2 + p(u^2 + v^2)^{p/2-1}[uu_{xx} + (u_x)^2 + vv_{xx} + (v_x)^2] \ldots (1)$$

Similarly, we can get $$\frac{\partial^2 \phi}{\partial y^2} = p\left(\frac{p}{2} - 1\right)(u^2 + v^2)^{p/2-2} \cdot 2(uu_y + vv_y)^2 + p(u^2 + v^2)^{p/2-1}[uu_{yy} + (u_y)^2 + vv_{yy} + (v_y)^2] \ldots (2)$$

Adding (1) and (2), we get

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = p\left(\frac{p}{2} - 1\right)(u^2 + v^2)^{p/2-2} \cdot 2\left[(uu_x + vv_x)^2 + (uu_y + vv_y)^2\right]$$

$$+ p(u^2 + v^2)^{p/2-1}[u(u_{xx} + u_{yy}) + v(v_{xx} + v_{yy}) + (u_x)^2 + (u_y)^2 + (v_x)^2 + (v_y)^2] \ldots (3)$$

Since $f(z)$ is analytic, $u$ and $v$ satisfy C-R equations and the Laplace equation,

$$u_x = v_y, \quad u_y = -v_x \quad \text{and} \quad u_{xx} + u_{yy} = 0 = v_{xx} + v_{yy} \ldots (4)$$
From (3) and (4), we get
\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = p \left( \frac{p}{2} - 1 \right) \left( u^2 + v^2 \right)^{\frac{p}{2} - 2} \cdot 2 \left[ (u u_x + v v_y)^2 + (-u v_x + v u_y)^2 \right] 
+ p \left( u^2 + v^2 \right)^{\frac{p}{2} - 1} \left[ u(0) + v(0) + (u_x)^2 + (v_y)^2 + (-u v_x + v u_y)^2 \right] 
= p \left( \frac{p}{2} - 1 \right) \left( u^2 + v^2 \right)^{\frac{p}{2} - 2} \cdot 2 \left( u^2 + v^2 \right)^{\left( u_x \right)^2 + (v_y)^2} + 2 p \left( u^2 + v^2 \right)^{\frac{p}{2} - 1} \left[ (u_x)^2 + (v_y)^2 \right] 
= 2 p \left( \frac{p}{2} - 1 + 1 \right) \left( u^2 + v^2 \right)^{\frac{p}{2} - 1} \left[ (u_x)^2 + (v_y)^2 \right] 
= p^2 \left( u^2 + v^2 \right)^{\frac{p}{2} - 1} \left[ (u_x)^2 + (v_y)^2 \right] 
= p^2 \left( u^2 + v^2 \right)^{\frac{p}{2} - 1} \left| f'(z) \right|^2 
\begin{align*}
\therefore \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left| f(z) \right|^p &= p^2 \left| f(z) \right|^{p-2} \left| f'(z) \right|^2 
\therefore \phi &= \left| f(z) \right|^p
\end{align*}
\]

**Example 13.8.** If \( f(z) \) is a regular function of \( z \), prove that \( \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \log |f(z)| = 0 \)

**Solution.** Suppose \( f(z) = u + iv \) is a regular function, then \( |f(z)| = \sqrt{u^2 + v^2} \) so that
\[
\log |f(z)| = \log \sqrt{u^2 + v^2} = \frac{1}{2} \log(u^2 + v^2) = \phi(x, y), \text{say}
\]
\[
\therefore \frac{\partial \phi}{\partial x} = \frac{u u_x + v v_y}{u^2 + v^2} \quad \text{and} \quad \frac{\partial^2 \phi}{\partial x^2} = \frac{(u^2 + v^2)[u u_{xx} + (u_x)^2 + v v_{yy} + (v_y)^2] - 2(u u_x + v v_y)^2}{(u^2 + v^2)^2} \quad \ldots (1)
\]
Similarly, we can get
\[
\frac{\partial^2 \phi}{\partial y^2} = \frac{(u^2 + v^2)[u u_{yy} + (u_y)^2 + v v_{xx} + (v_x)^2] - 2(u u_y + v v_x)^2}{(u^2 + v^2)^2} \quad \ldots (2)
\]
(1) + (2) gives
\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{(u^2 + v^2)[u u_{xx} + u_{xy} + (u_x)^2 + (u_y)^2 + v (v_x + v_y) + (v_x)^2 + (v_y)^2] - 2[(u u_x + v v_y)^2 + (u u_y + v v_x)^2]}{(u^2 + v^2)^2} \quad \ldots (3)
\]
Since \( f(z) \) is analytic, \( u \) and \( v \) satisfy C-R equations and the Laplace equation,
\[
u_x = v_y, \quad u_y = -v_x \quad \text{and} \quad u_{xx} + u_{yy} = 0 = v_{xx} + v_{yy} \ldots (4)
\]
From (3) and (4), we obtain
\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = \frac{(u^2 + v^2)[u(0) + (u_x)^2 + (v_x)^2 + v(0) + (v_y)^2 + (u_y)^2] - 2[(u u_x + v v_y)^2 + (u u_y + v v_x)^2]}{(u^2 + v^2)^2} 
= \frac{-2(u^2 + v^2)[(u_x)^2 + (v_y)^2] - 2(u^2 + v^2)[(u_x)^2 + (v_y)^2]}{(u^2 + v^2)^2} = 0
\]
\[
\therefore \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \log |f(z)| = 0 \quad \therefore \phi = \log |f(z)|
\]
Exercise 13

1. Find the orthogonal trajectories of the family curves \( x^4 + y^4 - 6x^2y^2 = c \)

2. Find the orthogonal trajectories of the family curves \( x^3y - xy^3 = c \)

3. If \( f(z) \) is an analytic function of \( z \), prove the following
   
   \[ (i) \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)|f(z)|^2 = 4|f'(z)|^2 \]
   
   \[ (ii) \left( \frac{\partial}{\partial x} |f(z)| \right)^2 + \left( \frac{\partial}{\partial y} |f(z)| \right)^2 = |f'(z)|^2 \]

Answers:

1. \( x^3y - xy^3 = d \)

2. \( x^4 + y^4 - 6x^2y^2 = d \)
Elementary functions

Exponential function of a complex variable: The exponential function of a complex variable \( z = x + iy \) is defined as \( e^z \) or \( \exp(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots \)

Properties:
1. \( e^z \) is an entire function
2. \( e^z \neq 0 \) for any \( z \)
3. \( e^z = e^{-z} \)
4. \( e^z \) is periodic function with an imaginary period \( 2\pi i \) \[ \therefore e^{z\pm 2\pi n i} = e^z \cdot e^{\pm 2\pi n i} = e^z, n \in \mathbb{Z}^+ \]

Circular functions of a complex variable: The circular function of a complex variable \( z = x + iy \) is defined as follows

1. \( \sin z = \frac{e^{iz} - e^{-iz}}{2i} \)
2. \( \cos z = \frac{e^{iz} + e^{-iz}}{2} \)
3. \( \tan z = \frac{\sin z}{\cos z} \)
4. \( \cosec z = \frac{1}{\sin z} \)
5. \( \sec z = \frac{1}{\cos z} \)
6. \( \cot z = \frac{\cos z}{\sin z} \)

Properties:
1. \( \sin z, \cos z \) are entire functions \[ \therefore e^z \text{ is entire} \]
2. \( \sin z, \cos z \) are periodic functions with period \( 2\pi \), while \( \tan z, \cot z \) have period \( \pi \).
3. \( \cos z, \sec z \) are even functions and \( \sin z, \tan z, \cosec z, \cot z \) are odd functions.
4. All the formulae for real circular functions are valid for complex circular functions.

Hyperbolic functions of a complex variable: The hyperbolic function of a complex variable \( z = x + iy \) is defined as follows

1. \( \sinh z = \frac{e^z - e^{-z}}{2} \)
2. \( \cosh z = \frac{e^z + e^{-z}}{2} \)
3. \( \tanh z = \frac{\sinh z}{\cosh z} \)
4. \( \cosech z = \frac{1}{\sinh z} \)
5. \( \sech z = \frac{1}{\cosh z} \)
6. \( \coth z = \frac{\cosh z}{\sinh z} \)

Properties:
1. \( \sinh z, \cosh z \) are entire functions \[ \therefore e^z \text{ is entire} \].
2. \( \sinh z, \cosh z \) are periodic functions having imaginary period \( 2\pi i \)
3. \( \cosh z \) is an even function while \( \sinh z \) is an odd function.
4. All the formulae for real circular functions are valid for complex circular functions.

5. \( \sinh 0 = \frac{e^0 - e^{-0}}{2} = 0; \quad \cosh 0 = \frac{e^0 + e^{-0}}{2} = 1; \quad \tanh 0 = \frac{\sinh 0}{\cosh 0} = 0 \)
Relation between circular functions and hyperbolic functions:

We know that \( \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \) and \( \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \)

Put \( \theta = iz \), we get

\[ \sin iz = \frac{e^{i(iz)} - e^{-i(iz)}}{2i} = -i \left( \frac{e^{-z} - e^{z}}{2} \right) = -i \frac{e^{-z} - e^{z}}{2} = i \sinh z \]

and \( \cos iz = \frac{e^{i(iz)} + e^{-i(iz)}}{2} = \frac{e^{-z} + e^{z}}{2} = \cosh z \)

Thus (i) \( \sin iz = i \sinh z \)  (ii) \( \cos iz = \cosh z \)  (iii) \( \tan iz = \frac{\sin iz}{\cos iz} = \frac{i \sinh z}{\cosh z} = i \tanh z \)

(iv) \( \sinh iz = \frac{e^{iz} - e^{-iz}}{2i} = i \left( \frac{e^{-z} - e^{z}}{2i} \right) = i \sin z \)  (v) \( \cosh iz = \frac{e^{iz} + e^{-iz}}{2} = \cos z \)

(iii) \( \tanh iz = \frac{\sinh iz}{\cosh iz} = \frac{i \sin z}{\cosh z} = i \tan z \)

Inverse hyperbolic functions: The inverse hyperbolic functions of a complex variable \( z = x + iy \) is defined as

1. \( \sinh^{-1} z = \log \left( z + \sqrt{z^2 + 1} \right) \)
2. \( \cosh^{-1} z = \log \left( z + \sqrt{z^2 - 1} \right) \)
3. \( \tanh^{-1} z = \frac{1}{2} \log \left( \frac{1 + z}{1 - z} \right) \)

Real and imaginary parts of elementary functions: The real part of an elementary function \( f(z) \) is denoted by \( \text{Re} \left( f(z) \right) \) and the imaginary part is denoted by \( \text{Im} \left( f(z) \right) \).

1. \( e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x (\cos y + i \sin y) \)
   \( \therefore \text{Re}(e^z) = e^x \cos y ; \quad \text{Im}(e^z) = e^x \sin y \)
2. \( \sin z = \sin(x + iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y \)
   \( \therefore \text{Re}(\sin z) = \sin x \cosh y ; \quad \text{Im}(\sin z) = \cos x \sinh y \)
3. \( \cos z = \cos(x + iy) = \cos x \cos iy - \sin x \sin iy = \cos x \cosh y - i \sin x \sinh y \)
   \( \therefore \text{Re}(\cos z) = \cos x \cosh y ; \quad \text{Im}(\cos z) = -\sin x \sinh y \)
4. \( \tan z = \frac{\sin z}{\cos z} = \frac{\sin(x+iy)}{\cos(x+iy)} = \frac{2 \sin(x+iy) \cos(x-iy)}{2 \cos(x+iy) \cos(x-iy)} \)
   \( = \frac{\sin(x+iy) + \sin(x+iy-x+iy)}{\cos(x+iy) + \cos(x+iy-x+iy)} = \frac{\sin 2x + \sin 2y}{\cos 2x + \cos 2y} \)
   \( = \frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y} = \frac{\sin 2x}{\cos 2x + \cosh 2y} + i \frac{\sinh 2y}{\cos 2x + \cosh 2y} \)
   \( \therefore \text{Re}(\tan z) = \frac{\sin 2x}{\cos 2x + \cosh 2y} ; \quad \text{Im}(\tan z) = \frac{\sinh 2y}{\cos 2x + \cosh 2y} \)
Properties: (i) $\sin \bar{z} = \overline{\sin z}$  
(ii) $\cos \bar{z} = \overline{\cos z}$  
(iii) $\tan \bar{z} = \frac{\overline{\sin z}}{\overline{\cos z}} = \overline{\tan z}$

5. $\sinh z = \frac{1}{2} \sin iz = -i \sin iz \quad (\therefore \sin iz = i \sinh z)$

Thus $\sinh z = -i \sin i(x + iy) = -i \sin(ix - y) = -i \sin(ix - y)$

$= -i(i\sin x\cos y - \cos i\sin y) = -i(i\sinh x\cos y - \cosh x\sin y)$

$= \sinh x\cos y + i\cosh x\sin y$

$\therefore \Re(\sinh z) = \sinh x\cos y; \quad \Im(\sinh z) = \cosh x\sin y$

6. $\cosh z = \cos iz = \cos i(x + iy) = \cos(ix - y)$

$= \cos ix\cos y + \sin ix\sin y = \cosh x\cos y + i\sinh x\sin y$

$\therefore \Re(\cosh z) = \cosh x\cos y; \quad \Im(\cosh z) = \sinh x\sin y$

7. $\text{sech } z = \frac{1}{\cosh z} = \frac{1}{\cos iz} \quad (\therefore \cosh z = \cos iz)$

$= \frac{1}{\cos i(x+iy)} = \frac{1}{\cos(ix-y)} = \frac{2\cos(ix+y)}{2\cos(ix-y)\cos(ix+y)}$

$= \frac{2(\cos ix\cos y - \sin ix\sin y)}{\cos(ix-y+ix+y) + \cos(ix-y-ix-y)}$

$= \frac{2(\cosh x\cos y - i\sinh x\sin y)}{\cos(2ix) + \cos(-2y)} = \frac{2\cosh x\cos y - 2i\sinh x\sin y}{\cosh 2x + \cosh 2y}$

$\therefore \Re(\text{sech } z) = \frac{2\cosh x\cos y}{\cosh 2x + \cosh 2y}; \quad \Im(\text{sech } z) = \frac{-2\sinh x\sin y}{\cosh 2x + \cosh 2y}$

Example 14.1. If $\cosh(u + iv) = x + iy$, prove that $\left(i\right) \frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = 1 \left(ii\right) \frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = 1$

Solution. Given $x + iy = \cosh(u + iv) = \cos i(u + iv) \quad \therefore \cosh z = \cos iz$

$= \cos(iu - v) = \cosh u\cos v + \sin iu\sin v$

$= \cosh u\cos v + i\sinh u\sin v$

Equating real and imaginary parts, we get $x = \cosh u\cos v; \quad y = \sinh u\sin v \ldots (1)$

(i) From (1), we get $\frac{x}{\cosh u} = \cos v \ldots (2)$ and $\frac{y}{\sinh u} = \sin v \ldots (3)$

Squaring (2) and (3) and adding, we get

$\frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = \cos^2 v + \sin^2 v = 1$

(ii) From (1), we get $\frac{x}{\cosh v} = \cosh u \ldots (4)$ and $\frac{y}{\sinh v} = \sinh u \ldots (5)$

Squaring (4) and (5) and subtracting, we get $\frac{x^2}{\cosh^2 u} - \frac{y^2}{\sinh^2 u} = \cosh^2 u - \sinh^2 u = 1$
Example 14.2. If \( \cosec \left( \frac{\pi}{4} + ix \right) = u + iv \), prove that \( (u^2 + v^2)^2 = 2(u^2 - v^2) \)

Solution. Given \( u + iv = \cosec \left( \frac{\pi}{4} + ix \right) = \frac{1}{\sin(\frac{\pi}{4} + ix)} = \frac{1}{\sin \frac{\pi}{4} \cos x + \cos \frac{\pi}{4} \sin x} \)

\[
= \frac{1}{\sqrt{2} \cosh x + isinh x} = \frac{\sqrt{2}(\cosh x - isinh x)}{(\cosh x + isinh x)(\cosh x - isinh x)}
\]

\[
= \frac{\sqrt{2}(\cosh x - isinh x)}{\cosh^2 x + sinh^2 x} = \frac{\sqrt{2} \cosh x}{\cosh 2x} - i \frac{\sqrt{2} \sinh x}{\cosh 2x}
\]

Equating real and imaginary parts, we get

\[ u = \frac{\sqrt{2} \cosh x}{\cosh 2x}; \quad v = -\frac{\sqrt{2} \sinh x}{\cosh 2x} \]

Now \( u^2 - v^2 = \left( \frac{\sqrt{2} \cosh x}{\cosh 2x} \right)^2 - \left( -\frac{\sqrt{2} \sinh x}{\cosh 2x} \right)^2 = \frac{2(\cosh^2 x - \sinh^2 x)}{\cosh^2 2x} = \frac{2}{\cosh^2 2x} \ldots (1) \)

Also \( u^2 + v^2 = \left( \frac{\sqrt{2} \cosh x}{\cosh 2x} \right)^2 + \left( -\frac{\sqrt{2} \sinh x}{\cosh 2x} \right)^2 = \frac{2(\cosh^2 x + \sinh^2 x)}{\cosh^2 2x} = \frac{2\cosh 2x}{\cosh^2 2x} \ldots (2) \)

From (1) and (2), we get \( (u^2 + v^2)^2 = 2(u^2 - v^2) \)

Logarithmic function of a complex variable: If \( z = x + iy \) and \( w = u + iv \) be two complex variables related such that \( w = e^z \) then \( z \) is said to be a logarithmic function of \( w \) and is written as \( z = \log w \).

General and principal values of logarithmic function: Since \( z = xe^{i\theta} \), we have \( \log z = \log(\text{re}^{i\theta}) = \log r + \log e^{i\theta} = \log r + \log e^{i(\theta + 2n\pi)} = \log r + i(\theta + 2n\pi), n \in \mathbb{Z} \)

Thus the general value of \( \log z \) is given by

\[ \log(x + iy) = \log \sqrt{x^2 + y^2} + i(\theta + 2n\pi), \quad (-\pi < \theta \leq \pi) \ldots (1) \]

where \( \theta = \begin{cases} \tan^{-1} \left( \frac{y}{x} \right), & x \geq 0, y \in \mathbb{R} \\ \pi + \tan^{-1} \left( \frac{y}{x} \right), & x < 0, y \geq 0 \\ -\pi + \tan^{-1} \left( \frac{y}{x} \right), & x < 0, y < 0 \end{cases} \)

Put \( n = 0 \) in (1), we get the principal value of \( \log z \) and is denoted by \( \text{Log} z \).

Thus \( \text{Log} z = \log \sqrt{x^2 + y^2} + i\theta, \quad (-\pi < \theta \leq \pi) \)

Example 14.3. Determine the general and principal values of \( (1 + i)^i \)

Solution. \( (1 + i)^i = e^{i \log(1 + i)} = e^{i \log 1} \)

Consider \( \log(1 + i) = \log \sqrt{1 + i \left( \tan^{-1} (1) + 2n\pi \right)} = \log \sqrt{2} + i \left( \frac{\pi}{4} + 2n\pi \right) \)

\[ \therefore (1 + i)^i = e^{i \log(1 + i)} = e^{i \log 1} = e^{i \left[ \log \sqrt{2} + i \left( \frac{\pi}{4} + 2n\pi \right) \right]} = e^{- \left( \frac{\pi}{4} + 2n\pi \right) + i \log \sqrt{2}} \]
\[
(1+i)^i = e^{-\left(\frac{\pi}{4} + 2n\pi\right)}e^{i\log \sqrt{2}} = e^{-\left(\frac{\pi}{4} + 2n\pi\right)}(\cos \log \sqrt{2} + i \sin \log \sqrt{2})
\]

Thus the general value of \((1+i)^i\) is \(e^{-\left(\frac{\pi}{4} + 2n\pi\right)}(\cos \log \sqrt{2} + i \sin \log \sqrt{2})\), \(n = 0, \pm 1, \pm 2, \ldots\) \hspace{1cm} (1)

The principal value of \((1+i)^i\) is \(e^{-\frac{\pi}{4}}(\cos \log \sqrt{2} + i \sin \log \sqrt{2})\) \hspace{1cm} [Put \(n = 0\) in (1)]

**Example 14.4.** Find the principal value of \(\sqrt{2i}\)

**Solution.** \(\sqrt{2i} = (2i)^{\frac{1}{2}} = e^{\log(2i)^{\frac{1}{2}}} = e^{\frac{1}{2}\log(2i)}\)

Consider \(\log(2i) = \log \sqrt{0+4} + i\left[\tan^{-1}\left(\frac{2}{0}\right) + 2n\pi\right] = \log 2 + i\left(\frac{\pi}{2} + 2n\pi\right)\)

\[
\therefore \sqrt{2i} = e^{\frac{1}{2}\log(2i)} = e^{\frac{1}{2}\left[\log 2 + i\left(\frac{\pi}{2} + 2n\pi\right)\right]} = e^{\frac{1}{2}\log 2 + i\left(\frac{\pi}{4} + n\pi\right)} = e^{\frac{1}{2}\log 2} \cdot e^{i\frac{\pi}{4}}
\]

Put \(n = 0\), we get \(\sqrt{2i} = e^{\frac{1}{2}\log 2} \cdot e^{i\frac{\pi}{4}} = \sqrt{2} \left(1 + i\right) = 1 + i\)

Thus the principal value of \(\sqrt{2i} = 1 + i\)

**Example 14.5.** If \(\tan \log(x + iy) = a + ib\), show that \(\tan \log(x^2 + y^2) = \frac{2a}{1-a^2-b^2}\), where \(a^2 + b^2 \neq 1\)

**Solution.** Put \(x = r \cos \theta, y = r \sin \theta\) so that \(r^2 = x^2 + y^2\) and \(\theta = \tan^{-1}\left(\frac{y}{x}\right)\)

\[a + ib = \tan \log(x + iy) = \tan \log(re^{i\theta}) = \tan((\log r + i\theta)) = \tan(u + iv)\]

where \(u = \log r = \log \sqrt{x^2 + y^2}\) and \(v = \theta\)

\[\therefore \tan(u + iv) = a + ib \hspace{1cm} \ldots (1)\]

Then \(\tan(u - iv) = a - ib \hspace{1cm} \ldots (2)\)

We know that \(\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}\)

Put \(A = u + iv, B = u - iv\)

\[\tan[(u + iv) + (u - iv)] = \frac{\tan(u + iv) + \tan(u - iv)}{1 - \tan(u + iv) \tan(u - iv)} = \frac{a + ib + a - ib}{1(\tan(u + iv)\tan(u - iv))} = \frac{2a}{1 - (a^2 + b^2)} \hspace{1cm} [\therefore \text{By (1) \& (2)}]\]

or \(\tan 2u = \frac{2a}{1 - (a^2 + b^2)}\) \hspace{1cm} or \(\tan 2 \log \sqrt{x^2 + y^2} = \frac{2a}{1 - a^2 - b^2}\)

or \(\tan \log(x^2 + y^2) = \frac{2a}{1 - a^2 - b^2}\), where \(a^2 + b^2 \neq 1\)

**Exercise**

1. If \(\sin(A + iB) = x + iy\), prove that (i) \(\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1\) \hspace{1cm} (ii) \(\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1\)

2. If \(\cos^{-1}(x + iy) = \alpha + i\beta\), show that
   \[i) \ x^2 \sec^2 \alpha - y^2 \cosec^2 \alpha = 1 \hspace{1cm} ii) \ x^2 \sech^2 \beta + y^2 \coth^2 \beta = 1\]
3. If \( \tan(A + iB) = x + iy, \) prove that (i) \( x^2 + y^2 + 2x \cot 2A = 1 \) (ii) \( x^2 + y^2 - 2y \coth 2B + 1 = 0 \)

4. Separate \( \tan^{-1}(x + iy) \) into real and imaginary parts

5. If \( \tan(\theta + i\phi) = e^{i\alpha} \), show that \( \theta = (n + \frac{1}{2}) \frac{\pi}{2} \) and \( \theta = \frac{1}{2} \log \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) \)

6. If \( u = \log \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) \), prove that (i) \( \tanh \frac{u}{2} = \tan \frac{\theta}{2} \) (ii) \( \theta = -i \log \tan\left(\frac{\pi}{4} + i \frac{u}{2}\right) \)

7. Separate \( \sin^{-1}(\cos \theta + i \sin \theta) \) into real and imaginary parts, where \( \theta \) is a positive acute angle.

8. Find the general and principal values of the following:
   (i) \( \log(1 + i\sqrt{3}) \) (ii) \( \log(-i) \) (iii) \( \log(-2) \) (iv) \( \log\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) \)

9. Prove that (i) \( i^i = e^{-(4n+1)\frac{\pi}{2}} \) (ii) \( \log i^i = -(4n+1)\frac{\pi}{2} \)

10. Prove that \( (\sqrt{i})^\alpha = e^{-\alpha} \cos \alpha \) where \( \alpha = \frac{\pi}{4\sqrt{2}} \)

11. Prove that \( \log\left(\frac{a+ib}{a-ib}\right) = 2i \tan^{-1}\left(\frac{b}{a}\right) \). Hence evaluate \( \cos\left[i \log\left(\frac{a+ib}{a-ib}\right)\right] \)

12. Prove that \( \tan\left[i \log\left(\frac{a+ib}{a-ib}\right)\right] = \frac{2ab}{a^2 - b^2} \).

13. If \( i^i = \infty \) then \( A + iB, \) prove that \( \tan\left(\frac{\pi A}{2}\right) = \frac{B}{A} \) and \( A^2 + B^2 = e^{-\pi B} \)

14. If \( (a + ib)^p = m^{x+iy} \), prove that one of the values of \( \frac{y}{x} \) is \( 2 \tan^{-1}\left(\frac{b}{a}\right) \div \log(a^2 + b^2) \)

15. Separate \( \log \sin(x + iy) \) into real and imaginary parts

16. Find all the roots of (i) \( \sin z = \cosh 4 \) (ii) \( \sinh z = i \)

17. Solve (i) \( \tanh z + 2 = 0 \) (ii) \( \cos z = 2 \)

**Solutions:**

4. Re[\( \tan^{-1}(x + iy) \)] = \( \frac{1}{2} \) tan\(^{-1}\left(\frac{2x}{1-x^2-y^2}\right) \), \( \text{Im}[\tan^{-1}(x + iy)] = \frac{1}{2} \tanh^{-1}\left(\frac{2y}{1+x^2+y^2}\right) \)

7. Re[\( \sin^{-1}(\cos \theta + i \sin \theta) \)] = \( \cos^{-1}\left(\sqrt{\sin \theta}\right) \), \( \text{Im}[\sin^{-1}(\cos \theta + i \sin \theta)] = \sin^{-1}\left(\sqrt{\sin \theta}\right) \)

8. (i) General Value = \( \log 2 + i\left(2n\pi + \frac{\pi}{3}\right) \); Principal Value = \( \log 2 + i\frac{\pi}{3} \)

   (ii) General Value = \( i\left(2n\pi - \frac{\pi}{3}\right) \); Principal Value = \( -i\frac{\pi}{2} \)

   (iii) General Value = \( \log 2 + i\left(\pi + 2n\pi\right) \) Principal Value \( \log 2 + i\pi \)

   (iv) General Value = \( i\left(2n\pi - \frac{3\pi}{4}\right) \); Principal Value = \( -i\frac{3\pi}{4} \)

15. Re[\( \log \sin(x + iy) \)] = \( \frac{1}{2} \log\left[\frac{1}{2}(\cosh 2y - \cos 2x)\right] \), \( \text{Im}[\log \sin(x + iy)] = \tan^{-1}(\cot x \tanh y) \)
16. (i) \( z = n\pi + (-1)^n \left( \frac{\pi}{2} - 4i \right); n \in \mathbb{Z} \)  
(ii) \( z = i \left( 2n\pi + \frac{\pi}{2} \right); n \in \mathbb{Z} \)

17. (i) \( z = -\log \sqrt{3} + i \left( 2n\pi + \frac{\pi}{2} \right); n \in \mathbb{Z} \)  
(ii) \( z = -i \log \left( 2 \pm \sqrt{3} \right) + 2n\pi; n \in \mathbb{Z} \)