Chapter 12 Solution to Problems

1. Find the exact altitude of a GPS satellite that has an orbital period equal to precisely one half of a sidereal day. Use a value of mean earth radius r_e = 6378.14 km and a sidereal day length of 23 hours 56 minutes 4.1 seconds.

Answer: The orbital period of the satellite is 11 hours 58 minutes 2.05 s = 43,082.05 s.
The orbital period is given by T where (Equation 2.6 squared on both sides)

$$T^2 = 4\,\pi^2\, a^3 / \mu$$

where $\mu = 3.986004418 \times 10^5$ km^3/s^2 and a is the radius of the orbit in km.
Hence

$$a^3 = T^2\mu / 4\,\pi^2 = 7.49602025 \times 10^{13} \text{ km}^3$$

$$a = 26{,}561.764 \text{ km}$$

The orbital altitude above a mean earth radius of 6378.14 km is 20,183.62 km.

2. Find the maximum Doppler shift of the L1 signal frequency for a GPS satellite at an altitude of 20,200 km when the satellite has an elevation angle of 10^o.
Hint: Maximum Doppler shift occurs when the observer is in the plane of the satellite orbit. Find the velocity of the satellite and the component of velocity towards the observer.

Answer: The velocity of the satellite in orbit is $2\,\pi\, a / T$ where T is the orbital period and a is the orbit radius. For a satellite with an orbit radius a = 20,200 + 6378.14 km = 26,578.14 km, the circumference of the orbit is $2\,\pi\, a$ = 166,995.38 km.

The orbital period is $T^2 = 4\,\pi^2\, a^3 / \mu$ where $\mu = 3.986004418 \times 10^5$ km^3/s^2.
Hence T = 43,121.90 s = 11 hrs 58 mins 41.8 s.
The velocity of the satellite is v_s = 166995.38 / 43121.90 = 3.87264 km/s.

We must calculate the relative velocity of the satellite towards an observer who is in the plane of the satellite orbit when the satellite has an elevation angle of 10^o. The geometry in the plane of the orbit is a triangle OGS, where O is the center of the earth, G is the observer at the earth's surface, and S is the satellite. When the satellite has an elevation angle of 10^o, the angle

OGS = $90^o + 10^o = 100^o$. The known lengths of the sides of the triangle are OG = r_e, OS = a, and the angle between the satellite velocity vector and the line OS is 90^o. We need to find the angle θ, between the satellite velocity vector and the line SG.

Denoting the angle GSO as α, we have

$$\sin\alpha / r_e = \sin 100^o / a$$

Hence

$$\sin\alpha = \sin 100^o \times r_e / a = 0.23633$$

$$\alpha = 13.670^o$$

The angle θ which defines the direction of the component of the satellite velocity towards the observer is given by

$$\theta = 90 - \alpha = 76.33$$

The component of velocity towards the observer is $v_r = v_s \cos\theta = 915.22$ m/s

The frequency of the L1 carrier signal for a GPS satellite is 1575.42 MHz, giving a wavelength of $\lambda = 0.190425$ m.

The maximum Doppler shift in the signal is

$$\Delta f = v_r / \lambda = 915.22 \text{ ms}^{-1} / 0.190425 \text{ m} = 4806.2 \text{ Hz}$$

A C/A code GPS receiver must be able to receive L1 signals that are shifted in frequency by up to 4.8 kHz. The shift will be an increase in frequency as the satellite approaches, falling to zero as the satellite passes overhead and then decreasing to a negative shift of – 4.8 kHz as the satellite reaches 10^o elevation before disappearing over the horizon.

3. An observer at the geographical north pole has a GPS receiver. At an instant in time, four GPS satellites all have the same range from the observer, and the GPS receiver records a measured delay time for the C/A signal of 0.17097528 s for each satellite. The four satellites' coordinates are calculated to be (0, -13280.5, 23002.5), (0, 13280.5, 23002.5), (-13280.5, 0, 23002.5), (13280.5, 0, 23002.5), where all distances are in km. Assuming an earth radius of 6378.0 km at the north pole, so that the observer's coordinates are (0,0, 6378), determine the clock offset error in the GPS receiver. (Use equations 12.1 and 12.3, and take the velocity of light in free space to be 2.99792458×10^8 m/s.)

Answer: From equation 12.1, the delay time of $T = 0.17097528$ s correspond to a psuedorange PR where

$$PR = T\,c = 0.17097528 \times 2.99792458 \times 10^8 = 51{,}257{,}099 \text{ m} = 51{,}257.099 \text{ km}$$

Equation 12.3 gives four simultaneous equations which give the psuedorange to the satellite

$$(X_i - U_x)^2 + (Y_i - U_y)^2 + (Z_i - U_z)^2 = (PR_i - \tau c)^2$$

where the receiver position is (U_x, U_y, U_z) and the four satellites have positions (X_i, Y_i, Z_i). The earth station location is known as (0, 0, 6378).

Hence the four simultaneous equations are (all distances in km)

$$(0 - 0)^2 + (13280.5 - 0)^2 + (23002.5 - 6378.0)^2 = (51{,}257.099 - \tau c)^2$$

$$(0 - 0)^2 + (-13280.5 - 0)^2 + (23002.5 - 6378.0)^2 = (51{,}257.099 - \tau c)^2$$

$$(13280.5 - 0)^2 + (0 - 0)^2 + (23002.5 - 6378.0)^2 = (51{,}257.099 - \tau c)^2$$

$$(-13280.5 - 0)^2 + (0 - 0)^2 + (23002.5 - 6378.0)^2 = (51{,}257.099 - \tau c)^2$$

Each of these equations gives the same result

$$13280.5^2 + (23002.5 - 6378.0)^2 = (51{,}257.099 - \tau c)^2$$

$$21{,}277.821 = (51{,}257.099 - \tau c)$$

Hence

$$\tau c = 29{,}979.278 \text{ km}$$

and the clock offset τ is

$$\tau = 0.10000010 = 100.000010 \text{ ms}$$

This simplified example illustrates how the four psuedorange equations can be solved to find clock offset error.

4. Accurate position location using GPS requires precise knowledge of the speed of light. In most applications, we use a velocity of light of 3.0×10^8 m/s. Solve Problem 3 above and then recalculate the clock offset using $c = 3 \times 10$ m/s instead of the more precise value given in Problem 3. What is the error in the clock offset? What is the difference in the ranges to the satellites when the approximate value for $c = 3 \times 10^8$ m/s is used? Discuss the corresponding position error due to the approximation. Why is it essential to use the exact value of the velocity of EM waves?

Answer: From equation 12.1, the delay time of $T = 0.17097528$ s correspond to a psuedorange PR where

$$PR = T\,c = 0.17097528 \times 2.99792458 \times 10^8 = 51{,}257.099 \text{ km}$$

Using the approximate speed of light as 3.0×10^8 m/s, the corresponding psuedorange is

$$PR = T\,c = 0.17097528 \times 3.0 \times 10^8 = 51{,}292{,}584 \text{ m} = 51{,}292.584 \text{ km}$$

Following the solution to Problem 3, the four range equations reduce to

$$13280.5^2 + (23002.5 - 6378.0)^2 = (51{,}292.584 - \tau c)^2$$

$$21{,}277.821 = (51{,}292.584 - \tau c)$$

or

$$\tau c = 30{,}014.763 \text{ km}$$

Using the approximate velocity of EM waves of 3×10^8 m/s, the clock offset is

$$\tau = 1.0004921 \times 10^{-1} \text{ s} = 100.04921 \text{ ms}$$

The correct value from Problem 3 is $\tau = 100.000010$ ms

The range to the satellite using $c = 3 \times 10^8$ m/s is

$R_{approx} = PR - \tau c = 51{,}292.584 - 30{,}014.763 = 21{,}312.821$ km when the approximate speed of light is used in the calculations. The solution to Problem 3 using the exact speed of light gives

$$R_{exact} = PR - \tau c = 51{,}257.099 - 29{,}979.277 = 21{,}277.821 \text{ km}$$

The difference in range between the exact and the approximate calculation is 35.00 km.

In the general case, the position calculation requires four ranges, so when the error in the range values has a random distribution and many measurements are averaged to obtain the final result, the resulting error in the calculation of the position of the GPS receiver, for DOP = 1, is $\sqrt{4} \times \Delta R = 2\,\Delta R$. However, in this case, the error in the range is the same for each range value, and we cannot average the error. The position error would be similar in magnitude to ΔR – about 35 km. This is clearly not an acceptable error, so we must use the exact value for the velocity of EM waves in all GPS calculations.

5. A C/A code GPS receiver is located at the geographic south pole, coordinates $(0,0, z_p)$. Four GPS satellites are used to determine the radius of the earth at the south pole. At the instant of time that the measurement is made, the satellites have coordinates

#1: (0, -13280.500, -23002.500) #2 : (0, 13280.500, -23002.500),

#3: (13280.500, 0, -23002.500) #4: (0, 0, -26561.000).

The corresponding measured delay times for the C/A code sequences from the satellites are

#1: 0.12102731 s, #2: 0.12102731 s #3: 0.12102731 s #4: 0.11738995 s

Find the clock offset in the GPS receiver, and determine the radius of the earth at the south pole. Use a value for the velocity of light in free space $c = 2.99792458 \times 10^8$ m/s, and work your solution to a precision of 1 m. You will need to solve two simultaneous non-linear equations from the set in Equation 12.3 in which the unknowns are the clock offset and the value of z_p. Start with an estimated value $z_p = 6378$ km, and then solve the two simultaneous equations. This will give two unequal values for the clock offset. Use iteration of the value of z_p to find the correct values for clock offset and earth radius at the south pole.

Answer: For satellites #1 through #3, the C/A code sequence delay time is T = 0.17097528 s correspond to a psuedorange PR where

$$PR_{1,2,3} = T_{1,2,3}\ c = 0.12102731 \times 2.99792458 \times 10^8 = 36{,}283.075 \text{ km}$$

For satellite #4, the delay time is T = 0.11738995 s and the psuedorange is

$$PR_4 = T_4\ c = 0.11738995 \times 2.99792458 \times 10^8 = 35{,}192.622 \text{ km}$$

Equation 12.3 gives four simultaneous equations which give the psuedorange to the satellite

$$(X_i - U_x)^2 + (Y_i - U_y)^2 + (Z_i - U_z)^2 = (PR_i - \tau c)^2$$

where the receiver position is (U_x, U_y, U_z) and the four satellites have positions (X_i, Y_i, Z_i). The earth station location at the south pole is known to be $(0, 0, z_p)$. Putting all distances in km and $c = 2.99792458 \times 10^5$ km/s:

The four simultaneous equations are

$$(0 - 0)^2 + (-13280.5 - 0)^2 + (-23002.5 - z_p)^2 = (36{,}283.075 - \tau c)^2$$

$$(-13280.5 - 0)^2 + (0 - 0)^2 + (-23002.5 - z_p)^2 = (36{,}283.075 - \tau c)^2$$

$$(+13280.5 - 0)^2 + (0 - 0)^2 + (-23002.5 - z_p)^2 = (36{,}283.075 - \tau c)^2$$

$$(0 - 0)^2 + (0 - 0)^2 + (-26561.000 - z_p)^2 = (35{,}192.622 - \tau c)^2$$

Each of first three equations gives the same result

$$13280.5^2 + (-23002.5 - z_p)^2 = (36{,}283.075 - \tau c)^2 \qquad (12.5.1)$$

The fourth equation gives

$$(-26561.0 - z_p)^2 = (35{,}192.622 - \tau c)^2 \qquad (12.5.2)$$

This is a pair of non-linear equations (because of the squares) with two unknowns, z_p and τ. There are several ways to solve such problems, but because we know the approximate value for z_p, iteration is one of the easiest, starting with an estimated value for z_p of - 6378 km.

Computer or hand calculator equation solving routines can also be used solve this problem. (Note: z_p is negative because the south pole is in the negative z direction for geocentric coordinates. The starting value of z_p = - 6378 km is the mean radius of the earth at the equator.)

Substituting z_p = - 6378 km in both equations and solving for τc, noting that taking square roots leads to two possible answers (± root), only one of which is valid:

From 12.5.1 τc = 15,005.253 km

From 12.5.2 τc = 15,009.622 km

The values of τc are not equal, with a difference of 4.369 km, so we must try another estimate.

Let's try z_p = - 6368 km.

From 12.5.1 τc = 14,999.002 km

From 12.5.2 τc = 15,001.622 km

The values of τc are now closer, at 2.62 km difference, and our revised estimate was in the correct direction, so we should try another estimate. Let's try z_p = - 6360 km.

From 12.5.1 τc = 14,991.187 km

From 12.5.2 τc = 14,911.621 km

A final trial with z_p = 6358 km gives

From 12.5.1 τc = 14,989.624 km

From 12.5.2 τc = 14,989.622 km

The difference is now 2 m, so we conclude that the radius of the earth at the south pole is 6358 km. Taking the mean of the two results above, the clock bias is

$$\tau = 14{,}989.623 / c = 0.050000 \text{ seconds} = 50.0000\text{ms}.$$

The values for the range error agree within 2 m, so we can find the clock offset error with considerable confidence. This example illustrates how a GPS receiver can calculate clock offset error and true range to the satellites within one meter, using the solution of simultaneous non-linear equations. This example is greatly simplified to make it possible to obtain a solution by hand calculation.